A Beginner's Course in Boundary Element Methods

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Universal Publishers
Boca Raton, Florida

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Universal Publishers
Boca Raton, Florida • USA
2007

ISBN: 1-58112-974-2
13-ISBN: 978-1-58112-974-8

www.universal-publishers.com

Preface

During the last few decades, the boundary element method, also known as the boundary integral equation method or boundary integral method, has gradually evolved to become one of the few widely used numerical techniques for solving boundary value problems in engineering and physical sciences. In implementing the method, only the boundary of the solution domain has to be discretized into elements. In the case of a two-dimensional problem, this is really easy to do. Put closely packed points on the boundary (a curve) and join up two consecutive neighboring points to form straight line elements.

In March 1985, when I started research work for a doctoral degree in the Department of Applied Mathematics at the University of Adelaide, Australia, I was introduced to the method by my thesis supervisor, David L Clements. At that time, the term "boundary element method" was relatively new. It was apparently first used in a 1977 paper by CA Brebbia and J Dominguez[1]. Carlos Brebbia and his co-researchers had undoubtedly played an important role in introducing the method to the engineering research community. Apparently less than 200 journal papers whose titles contained the term "boundary element method" could be found in 1985. In 2006, there were several thousand or perhaps even more such papers.

The history of the method could, however, be traced back to an earlier time, well before the 1970s. The mathematics that laid the theoretical foundation for the development of the method could be found in the works of famous mathematicians like Laplace, Gauss, Green, Fredholm, Betti, Somigliana, Muskhelishvili, Mikhlin and Kupradze. In the 1960s, there were attempts at using electronic computers to approximate solutions of potential problems through the use of boundary integral equations, notably the pioneering works of MA Jaswon and

[1] CA Brebbia and J Dominguez, "Boundary element methods for potential problems," *Applied Mathematical Modelling*, Volume 1, 1977, pp. 372-378.

GT Symm[2]. The work of Frank J Rizzo[3] was regarded by many researchers as the beginning of a novel direct boundary integral method for the numerical solution of elasticity problems.

After completing my doctoral work in the middle of 1987, I continued to keep myself informed on the development of the boundary integral method and related mathematical works, pick up some new ideas now and then, attend conferences, give talks and seminars, and contribute to boundary element research with applications to problems in engineering and physical sciences. Some specific research areas I had worked on using the boundary integral method include linear fracture mechanics (accurate computation of stress intensity factors using special Green's functions), analyses of nonhomogeneous media (such as functionally graded materials), diffusion with specification of mass, modeling of photonic crystal fibers, integral formulation of imperfect interfaces and bioheat transfer.

Sometimes, I undertook the task of introducing the method to beginners, mainly advanced undergraduate and research students who were working on projects under my supervision. To do this, I had produced various notes over a period of time. The chapters in this book were written based on those notes. In writing this book, I assume that the readers have some prior basic knowledge of vector calculus (covering topics such as line, surface and volume integrals and the various integral theorems), ordinary and partial differential equations, complex variables and computer programming.

FORTRAN 77 codes for the numerical procedures discussed are listed in the chapters[4]. Some justifications, if any is needed

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[2]One may refer to the following papers: (a) MA Jaswon, "An integral equation method in potential theory I," *Proceedings of the Royal Society of London Series A*, Volume 275, 1963, pp. 23-32, and (b) GT Symm, "An integral equation method in potential theory II," *Proceedings of the Royal Society of London Series A*, Volume 275, 1963, pp. 33-46.

[3]FJ Rizzo, "An integral equation approach to boundary value problems of classical elastostatics," *Quarterly of Applied Mathematics*, Volume 25, 1967, pp. 83-95. This was the work presented by FJ Rizzo in his doctoral dissertation. Much later on in 1993, it won him an ASME Warner Medal.

[4]Readers who are interested in obtaining the codes in electronic form may e-mail me at mwtang@ntu.edu.sg.

at all, for using good old FORTRAN 77 would be as follows. Firstly, in spite of its seniority, it still remains a powerful "number crunching tool". Secondly, its codes are relatively easy to decipher and would be of some use even to readers who are attempting to implement the numerical procedures using newer software tools (such as C++ and MATLAB). Thirdly, free FORTRAN 77 compilers (e.g. FTN77 from Salford Software and GNU Fortran) can be downloaded from the internet.

The constant encouragement and support of my dear wife, Young-Soon, had greatly motivated me to start and finish writing this book. I would like to thank Ean-Hin Ooi, Lukito Jayaputra, Bao Ing Yun, Jackson R Jones, Joris Vankerschaver and Alessandro Vaccari for informing me of errors in an earlier version of this book and Jeff Young and Rebekah Galy of Universal Publishers for their prompt replies to all my questions on the publication of this book.

Whye-Teong Ang, Singapore, 2007, 2014

Contents

Chapter 1

Two-dimensional Laplace's Equation

1.1 Introduction

Perhaps a good starting point for introducing boundary element methods is through solving boundary value problems governed by the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{1.1}$$

The Laplace's equation occurs in the formulation of problems in many diverse fields of studies in engineering and physical sciences, such as thermostatics, elastostatics, electrostatics, magnetostatics, ideal fluid flow and flow in porous media.

An interior boundary value problem which is of practical interest requires solving Eq. (1.1) in the two-dimensional region R (on the Oxy plane) bounded by a simple closed curve C subject to the boundary conditions

$$\begin{aligned} \phi &= f_1(x, y) \text{ for } (x, y) \in C_1, \\ \frac{\partial\phi}{\partial n} &= f_2(x, y) \text{ for } (x, y) \in C_2, \end{aligned} \tag{1.2}$$

where f_1 and f_2 are suitably prescribed functions and C_1 and C_2 are non-intersecting curves such that $C_1 \cup C_2 = C$. Refer to Figure 1.1 for a geometrical sketch of the problem.

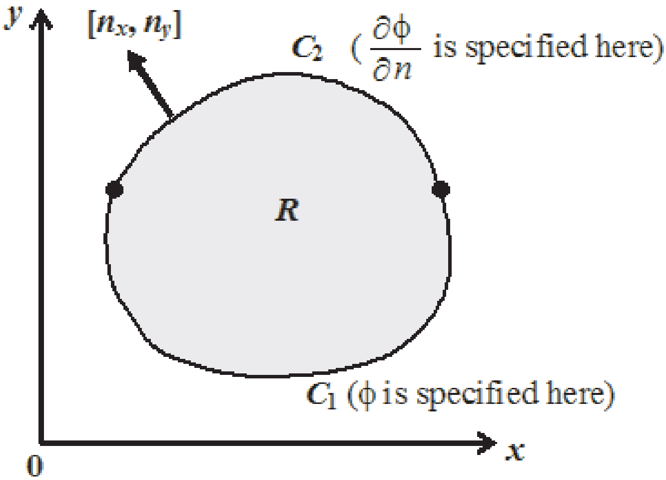

Figure 1.1

The normal derivative $\partial\phi/\partial n$ in Eq. (1.2) is defined by

$$\frac{\partial \phi}{\partial n} = n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y}, \tag{1.3}$$

where n_x and n_y are respectively the x and y components of a unit normal vector to the curve C. Here the unit normal vector $[n_x, n_y]$ on C is taken to be pointing away from the region R. Note that the normal vector may vary from point to point on C. Thus, $[n_x, n_y]$ is a function of x and y.

The boundary conditions given in Eq. (1.2) are assumed to be properly posed so that the boundary value problem has a unique solution, that is, it is assumed that one can always find a function $\phi(x, y)$ satisfying Eqs. (1.1)-(1.2) and that there is only one such function.

For a particular example of practical situations involving the boundary value problem above, one may mention the classical heat conduction problem where ϕ denotes the steady-state temperature in an isotropic solid. Eq. (1.1) is then the temperature governing equation derived, under certain assumptions, from the law of conservation of heat energy together with the

Fourier's heat flux model. The heat flux out of the region R across the boundary C is given by $-\kappa\partial\phi/\partial n$, where κ is the thermal heat conductivity of the solid. Thus, the boundary conditions in Eq. (1.2) imply that at each and every given point on C either the temperature or the heat flux (but not both) is known. To determine the temperature field in the solid, one has to solve Eq. (1.1) in R to find the solution that satisfies the prescribed boundary conditions on C.

In general, it is difficult (if not impossible) to solve exactly the boundary value problem defined by Eqs. (1.1)-(1.2). The mathematical complexity involved depends on the geometrical shape of the region R and the boundary conditions given in Eq. (1.2). Exact solutions can only be found for relatively simple geometries of R (such as a square region) together with particular boundary conditions. For more complicated geometries or general boundary conditions, one may have to resort to numerical (approximate) techniques for solving Eqs. (1.1)-(1.2).

This chapter introduces a boundary element method for the numerical solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2). We show how a boundary integral solution can be derived for Eq. (1.1) and applied to obtain a simple boundary element procedure for approximately solving the boundary value problem under consideration. The implementation of the numerical procedure on the computer, achieved through coding in FORTRAN 77, is discussed in detail.

1.2 Fundamental Solution

If we use polar coordinates r and θ centered about $(0,0)$, as defined by $x = r\cos\theta$ and $y = r\sin\theta$, and introduce $\psi(r,\theta) = \phi(r\cos\theta, r\sin\theta)$, we can rewrite Eq. (1.1) as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0. \tag{1.4}$$

For the case in which ψ is independent of θ, that is, if ψ is a function of r alone, Eq. (1.4) reduces to the ordinary differential equation

$$\frac{d}{dr}(r\frac{d}{dr}[\psi(r)]) = 0 \text{ for } r \neq 0. \tag{1.5}$$

The ordinary differential equation in Eq. (1.5) can be easily integrated twice to yield the general solution

$$\psi(r) = A\ln(r) + B, \tag{1.6}$$

where A and B are arbitrary constants.

From (1.6), it is obvious that the two-dimensional Laplace's equation in Eq. (1.1) admits a class of particular solutions given by

$$\phi(x,y) = A\ln\sqrt{x^2+y^2} + B \quad \text{for } (x,y) \neq (0,0). \tag{1.7}$$

If we choose the constants A and B in (1.7) to be $1/(2\pi)$ and 0 respectively and shift the center of the polar coordinates from $(0,0)$ to the general point (ξ,η), a particular solution of Eq. (1.1) is

$$\phi(x,y) = \frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2} \quad \text{for } (x,y) \neq (\xi,\eta). \tag{1.8}$$

As we shall see, the particular solution in Eq. (1.8) plays an important role in the development of boundary element methods for the numerical solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2). We specially denote this particular solution using the symbol $\Phi(x,y;\xi,\eta)$, that is, we write

$$\Phi(x,y;\xi,\eta) = \frac{1}{4\pi}\ln[(x-\xi)^2+(y-\eta)^2]. \tag{1.9}$$

We refer to $\Phi(x,y;\xi,\eta)$ in Eq. (1.9) as the fundamental solution of the two-dimensional Laplace's equation. Note that $\Phi(x,y;\xi,\eta)$ satisfies Eq. (1.1) everywhere except at (ξ,η) where it is not well defined.

1.3 Reciprocal Relation

If ϕ_1 and ϕ_2 are any two solutions of Eq. (1.1) in the region R bounded by the simple closed curve C then it can be shown that

$$\int_C (\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n}) ds(x,y) = 0. \tag{1.10}$$

Eq. (1.10) provides a reciprocal relation between any two solutions of the Laplace's equation in the region R bounded by the curve C. It may be derived from the two-dimensional version of the Gauss-Ostrogradskii (divergence) theorem as explained below.

According to the divergence theorem, if $\underline{\mathbf{F}}= u(x,y)\underline{\mathbf{i}}+v(x,y)\underline{\mathbf{j}}$ is a well defined vector function such that $\underline{\nabla}\cdot\underline{\mathbf{F}} = \partial u/\partial x+\partial v/\partial y$ exists in the region R bounded by the simple closed curve C then

$$\int_C \underline{\mathbf{F}} \cdot \underline{\mathbf{n}}\, ds(x,y) = \iint_R \underline{\nabla} \cdot \underline{\mathbf{F}}\, dxdy,$$

that is,

$$\int_C [un_x + vn_y] ds(x,y) = \iint_R [\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}] dxdy,$$

where $\underline{\mathbf{n}} = [n_x, n_y]$ is the unit normal vector to the curve C, pointing away from the region R.

Since ϕ_1 and ϕ_2 are solutions of Eq. (1.1), we may write

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} &= 0, \\ \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} &= 0. \end{aligned}$$

If we multiply the first equation by ϕ_2 and the second one by ϕ_1 and take the difference of the resulting equations, we obtain

$$\frac{\partial}{\partial x}(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x})+\frac{\partial}{\partial y}(\phi_2\frac{\partial\phi_1}{\partial y}-\phi_1\frac{\partial\phi_2}{\partial y})=0,$$

which can be integrated over R to give

$$\iint\limits_R[\frac{\partial}{\partial x}(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x})+\frac{\partial}{\partial y}(\phi_2\frac{\partial\phi_1}{\partial y}-\phi_1\frac{\partial\phi_2}{\partial y})]dxdy=0.$$

Application of the divergence theorem to convert the double integral over R into a line integral over C yields

$$\int\limits_C[(\phi_2\frac{\partial\phi_1}{\partial x}-\phi_1\frac{\partial\phi_2}{\partial x})n_x+(\phi_2\frac{\partial\phi_1}{\partial y}-\phi_1\frac{\partial\phi_2}{\partial y})n_y]ds(x,y)=0$$

which is essentially Eq. (1.10).

Together with the fundamental solution given by Eq. (1.9), the reciprocal relation in Eq. (1.10) can be used to derive a useful boundary integral solution for the two-dimensional Laplace's equation.

1.4 Boundary Integral Solution

Let us take $\phi_1=\Phi(x,y;\xi,\eta)$ (the fundamental solution as defined in Eq. (1.9)) and $\phi_2=\phi$, where ϕ is the required solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2).

Since $\Phi(x,y;\xi,\eta)$ is not well defined at the point (ξ,η), the reciprocal relation in Eq. (1.10) is valid for $\phi_1=\Phi(x,y;\xi,\eta)$ and $\phi_2=\phi$ only if (ξ,η) does not lie in the region $R\cup C$. Thus,

$$\begin{aligned}&\int\limits_C[\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta))\\&-\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y)\\=\;&0\text{ for }(\xi,\eta)\notin R\cup C.\end{aligned}\qquad(1.11)$$

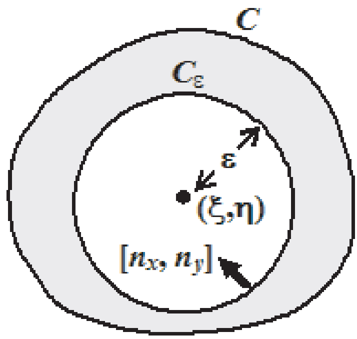

Figure 1.2

A more interesting and useful integral equation than Eq. (1.11) can be derived from Eq. (1.10) if we take the point (ξ, η) to lie in the region $R \cup C$.

For the case in which (ξ, η) lies in the interior of R, Eq. (1.10) is valid if we replace C by $C \cup C_\varepsilon$, where C_ε is a circle of center (ξ, η) and radius ε as shown in Figure 1.2[1]. This is because $\Phi(x, y; \xi, \eta)$ and its first order partial derivatives (with respect to x or y) are well defined in the region between C and C_ε. Thus, for C and C_ε in Figure 1.2, we can write

$$\begin{aligned} & \int\limits_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ & -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ & +\int\limits_{C_\varepsilon} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ & -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ = \; & 0. \end{aligned} \tag{1.12}$$

[1]The divergence theorem is not only applicable for simply connected regions but also for multiply connected ones such as the one shown in Figure 1.2. For the region in Figure 1.2, the unit normal vector to C_ε (the inner boundary) points towards the center of the circle.

Eq. (1.12) holds for any radius $\varepsilon > 0$, so long as the circle C_ε (in Figure 1.2) lies completely inside the region bounded by C. Thus, we may let $\varepsilon \to 0^+$ in Eq. (1.12). This gives

$$\begin{aligned} & \int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ & -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ = & -\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ & -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y). \end{aligned} \tag{1.13}$$

Using polar coordinates r and θ centered about (ξ,η) as defined by $x-\xi = r\cos\theta$ and $y-\eta = r\sin\theta$, we may write

$$\begin{aligned} \Phi(x,y;\xi,\eta) &= \frac{1}{2\pi}\ln(r), \\ \frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)] &= n_x\frac{\partial}{\partial x}[\Phi(x,y;\xi,\eta)] + n_y\frac{\partial}{\partial y}[\Phi(x,y;\xi,\eta)] \\ &= \frac{n_x\cos\theta + n_y\sin\theta}{2\pi r}. \end{aligned} \tag{1.14}$$

The Taylor's series of $\phi(x,y)$ about the point (ξ,η) is given by

$$\phi(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m \phi}{\partial x^k \partial y^{m-k}})\bigg|_{(x,y)=(\xi,\eta)} \frac{(x-\xi)^k(y-\eta)^{m-k}}{k!(m-k)!}.$$

On the circle C_ε, $r=\varepsilon$. Thus,

$$\begin{aligned} \phi(x,y) = & \sum_{m=0}^{\infty}\sum_{k=0}^{m}(\frac{\partial^m}{\partial x^k \partial y^{m-k}}[\phi(x,y)])\bigg|_{(x,y)=(\xi,\eta)} \\ & \times\frac{\varepsilon^m\cos^k\theta\sin^{m-k}\theta}{k!(m-k)!} \text{ for } (x,y)\in C_\varepsilon. \end{aligned} \tag{1.15}$$

Similarly, we may write

$$\frac{\partial}{\partial n}[\phi(x,y)] = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m}{\partial x^k \partial y^{m-k}}\{\frac{\partial}{\partial n}[\phi(x,y)]\}\right)\Big|_{(x,y)=(\xi,\eta)} \times \frac{\varepsilon^m \cos^k\theta \sin^{m-k}\theta}{k!(m-k)!} \quad \text{for } (x,y)\in C_\varepsilon. \tag{1.16}$$

Using Eqs. (1.14), (1.15) and (1.16) and writing $ds(x,y) = \varepsilon d\theta$ with θ ranging from 0 to 2π, we may now attempt to evaluate the limit on the right hand side of Eq. (1.13). On C_ε, the normal vector $[n_x, n_y]$ is given by $[-\cos\theta, -\sin\theta]$. Thus,

$$\begin{aligned}
&\int\limits_{C_\varepsilon} \phi(x,y)\frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta]ds(x,y) \\
= \quad & -\frac{1}{2\pi}\phi(\xi,\eta)\int\limits_0^{2\pi} d\theta \\
& -\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{k=0}^{m}\frac{\varepsilon^m}{k!(m-k)!}\left(\frac{\partial^m\phi}{\partial x^k\partial y^{m-k}}\right)\Big|_{(x,y)=(\xi,\eta)} \\
& \times\int\limits_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta \\
\to \quad & -\phi(\xi,\eta) \quad \text{as } \varepsilon\to 0^+,
\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}
&\int\limits_{C_\varepsilon} \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}[\phi(x,y)]ds(x,y) \\
= \quad & \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\frac{\partial^m}{\partial x^k\partial y^{m-k}}\left(\frac{\partial}{\partial n}[\phi(x,y)]\right)\right)\Big|_{(x,y)=(\xi,\eta)} \\
& \times\frac{\varepsilon^{m+1}\ln(\varepsilon)}{k!(m-k)!}\int\limits_0^{2\pi}\cos^k\theta\sin^{m-k}\theta d\theta \\
\to \quad & 0 \quad \text{as } \varepsilon\to 0^+,
\end{aligned} \tag{1.18}$$

since $\varepsilon^{m+1}\ln(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ for $m = 0, 1, 2, \cdots$.

Consequently, as $\varepsilon \to 0^+$, Eq. (1.13) yields

$$\begin{aligned}\phi(\xi,\eta) &= \int\limits_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ &\qquad -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ &\qquad \text{for } (\xi,\eta) \in R. \qquad (1.19)\end{aligned}$$

Together with Eq. (1.9), Eq. (1.19) provides us with a boundary integral solution for the two-dimensional Laplace's equation. If both ϕ and $\partial\phi/\partial n$ are known at all points on C, the line integral in Eq. (1.19) can be evaluated (at least in theory) to calculate ϕ at any point (ξ,η) in the interior of R. From the boundary conditions (1.2), however, at any given point on C, either ϕ or $\partial\phi/\partial n$, not both, is known.

To solve the interior boundary value problem, we must find the unknown ϕ and $\partial\phi/\partial n$ on C_2 and C_1 respectively. As we shall see later on, this may be done through manipulation of data on the boundary C only, if we can derive a boundary integral formula for $\phi(\xi,\eta)$, similar to the one in Eq. (1.19), for a general point (ξ,η) that lies on C.

For the case in which the point (ξ,η) lies on C, Eq. (1.10) holds if we replace the curve C by $D \cup D_\varepsilon$, where the curves D and D_ε are as shown in Figure 1.3. (If C_ε is the circle of center (ξ,η) and radius ε, then D is the part of C that lies outside C_ε and D_ε is the part of C_ε that is inside R.) Thus,

$$\begin{aligned}&\int\limits_D [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ &\qquad -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\ &= -\int\limits_{D_\varepsilon} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\ &\qquad -\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y). \qquad (1.20)\end{aligned}$$

Let us examine what happens to Eq. (1.20) when we let $\varepsilon \to 0^+$.

As $\varepsilon \to 0^+$, the curve D tends to C. Thus, we may write

$$\begin{aligned}
&\int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\
&-\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y) \\
= &-\lim_{\varepsilon\to 0^+}\int_{D_\varepsilon} [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\
&-\Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y). \qquad (1.21)
\end{aligned}$$

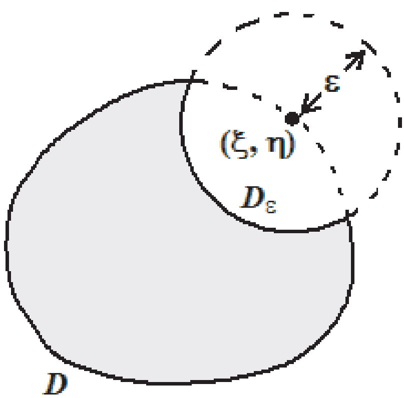

Figure 1.3

Note that, unlike in Eq. (1.13), the line integral over C in Eq. (1.21) is improper as its integrand is not well defined at (ξ,η) which lies on C. Strictly speaking, the line integration should be over the curve C without an infinitesimal segment that contains the point (ξ,η), that is, the line integral over C in Eq. (1.21) has to be interpreted in the Cauchy principal sense if (ξ,η) lies on C.

To evaluate the limit on the right hand side of Eq. (1.21), we need to know what happens to D_ε when we let $\varepsilon \to 0^+$. Now if (ξ, η) lies on a smooth part of C (not at where the gradient of the curve changes abruptly, that is, not at a corner point, if there is any), one can intuitively see that the part of C inside C_ε approaches an infinitesimal straight line as $\varepsilon \to 0^+$. Thus, we expect D_ε to tend to a semi-circle as $\varepsilon \to 0^+$, if (ξ, η) lies on a smooth part of C. It follows that in attempting to evaluate the limit on the right hand side of Eq. (1.21) we have to integrate over only half a circle (instead of a full circle as in the case of Eq. (1.13)).

Modifying Eqs. (1.17) and (1.18), we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int\limits_{D_\varepsilon} \phi(x,y) \frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)] ds(x,y) &= -\frac{1}{2}\phi(\xi,\eta), \\
\lim_{\varepsilon \to 0^+} \int\limits_{D_\varepsilon} \Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}[\phi(x,y)] ds(x,y) &= 0.
\end{aligned}$$

Hence Eq. (1.21) gives

$$\begin{aligned}
\frac{1}{2}\phi(\xi,\eta) = &\int\limits_{C} [\phi(x,y) \frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) \\
&-\Phi(x,y;\xi,\eta) \frac{\partial}{\partial n}(\phi(x,y))] ds(x,y) \\
&\text{for } (\xi,\eta) \text{ lying on a smooth part of } C.
\end{aligned} \tag{1.22}$$

Together with the boundary conditions in Eq. (1.2), Eq. (1.22) may be applied to obtain a numerical procedure for determining the unknown ϕ and/or $\partial\phi/\partial n$ on the boundary C. Once ϕ and $\partial\phi/\partial n$ are known at all points on C, the solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2) is given by Eq. (1.19) at any point (ξ, η) inside R. More details are given in Section 1.5 below.

For convenience, we may write Eqs. (1.11), (1.19) and (1.22) as a single equation given by

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int_C [\phi(x,y)\frac{\partial}{\partial n}(\Phi(x,y;\xi,\eta)) - \Phi(x,y;\xi,\eta)\frac{\partial}{\partial n}(\phi(x,y))]ds(x,y), \quad (1.23)$$

if we define

$$\lambda(\xi,\eta) = \begin{cases} 0 & \text{if } (\xi,\eta) \notin R \cup C, \\ 1/2 & \text{if } (\xi,\eta) \text{ lies on a smooth part of } C, \\ 1 & \text{if } (\xi,\eta) \in R. \end{cases} \quad (1.24)$$

1.5 Boundary Element Solution with Constant Elements

We now show how Eq. (1.23) may be applied to obtain a simple boundary element procedure for solving numerically the interior boundary value problem defined by Eqs. (1.1)-(1.2).

The boundary C is discretized into N very small straight line segments $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$, that is,

$$C \simeq C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)}. \quad (1.25)$$

The sides or the boundary elements $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$ are constructed as follows. We put N well spaced out points given by $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $\cdots$, $(x^{(N-1)}, y^{(N-1)})$ and $(x^{(N)}, y^{(N)})$ on C, in the order given, following the counter clockwise direction. Defining $(x^{(N+1)}, y^{(N+1)}) = (x^{(1)}, y^{(1)})$, we take $C^{(k)}$ to be the boundary element from $(x^{(k)}, y^{(k)})$ to $(x^{(k+1)}, y^{(k+1)})$ for $k = 1, 2, \cdots , N$.

As an example, in Figure 1.4, the boundary $C = C_1 \cup C_2$ in Figure 1.1 is approximated using 5 boundary elements denoted by $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ and $C^{(5)}$.

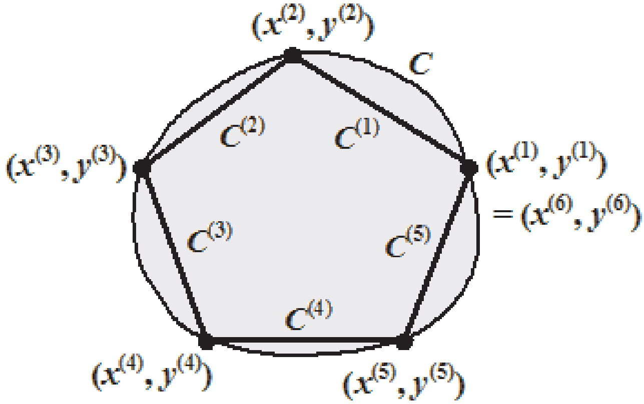

Figure 1.4

For a simple approximation of ϕ and $\partial\phi/\partial n$ on the boundary C, we assume that these functions are constants over each of the boundary elements. Specifically, we make the approximation:

$$\phi \simeq \overline{\phi}^{(k)} \text{ and } \frac{\partial\phi}{\partial n} = \overline{p}^{(k)} \text{ for } (x,y) \in C^{(k)} \ (k = 1, 2, \cdots, N), \tag{1.26}$$

where $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ are respectively the values of ϕ and $\partial\phi/\partial n$ at the midpoint of $C^{(k)}$.

With Eqs. (1.25) and (1.26), we find that Eq. (1.23) can be approximately written as

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \sum_{k=1}^{N}\{\overline{\phi}^{(k)}\mathcal{F}_2^{(k)}(\xi,\eta) - \overline{p}^{(k)}\mathcal{F}_1^{(k)}(\xi,\eta)\}, \tag{1.27}$$

where

$$\begin{aligned}\mathcal{F}_1^{(k)}(\xi,\eta) &= \int\limits_{C^{(k)}} \Phi(x,y;\xi,\eta)ds(x,y),\\ \mathcal{F}_2^{(k)}(\xi,\eta) &= \int\limits_{C^{(k)}} \frac{\partial}{\partial n}[\Phi(x,y;\xi,\eta)]ds(x,y).\end{aligned} \tag{1.28}$$

For a given k, either $\overline{\phi}^{(k)}$ or $\overline{p}^{(k)}$ (not both) is known from the boundary conditions in Eq. (1.2). Thus, there are N unknown constants on the right hand side of Eq. (1.27). To determine their values, we have to generate N equations containing the unknowns.

If we let (ξ, η) in Eq. (1.27) be given in turn by the midpoints of $C^{(1)}$, $C^{(2)}$, $\cdots$, $C^{(N-1)}$ and $C^{(N)}$, we obtain

$$\frac{1}{2}\overline{\phi}^{(m)} = \sum_{k=1}^{N}\{\overline{\phi}^{(k)}\mathcal{F}_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) - \overline{p}^{(k)}\mathcal{F}_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)})\}$$
$$\text{for } m = 1, 2, \cdots, N, \qquad (1.29)$$

where $(\overline{x}^{(m)}, \overline{y}^{(m)})$ is the midpoint of $C^{(m)}$.

In the derivation of Eq. (1.29), we take $\lambda(\overline{x}^{(m)}, \overline{y}^{(m)}) = 1/2$, since $(\overline{x}^{(m)}, \overline{y}^{(m)})$ being the midpoint of $C^{(m)}$ lies on a smooth part of the approximate boundary $C^{(1)} \cup C^{(2)} \cup \cdots \cup C^{(N-1)} \cup C^{(N)}$.

Eq. (1.29) constitutes a system of N linear algebraic equations containing the N unknowns on the right hand side of Eq. (1.27). We may rewrite it as

$$\sum_{k=1}^{N} a^{(mk)} z^{(k)} = \sum_{k=1}^{N} b^{(mk)} \text{ for } m = 1, 2, \cdots, N, \qquad (1.30)$$

where $a^{(mk)}$, $b^{(mk)}$ and $z^{(k)}$ are defined by

$$a^{(mk)} = \begin{cases} -\mathcal{F}_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) \text{ if } \phi \text{ is specified over } C^{(k)}, \\ \mathcal{F}_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) - \frac{1}{2}\delta^{(mk)} \text{ if } \partial\phi/\partial n \text{ is} \\ \qquad\qquad \text{specified over } C^{(k)}, \end{cases}$$

$$b^{(mk)} = \begin{cases} \overline{\phi}^{(k)}(-\mathcal{F}_2^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) + \frac{1}{2}\delta^{(mk)}) \\ \qquad\qquad \text{if } \phi \text{ is specified over } C^{(k)}, \\ \overline{p}^{(k)}\mathcal{F}_1^{(k)}(\overline{x}^{(m)}, \overline{y}^{(m)}) \text{ if } \partial\phi/\partial n \text{ is specified} \\ \qquad\qquad \text{over } C^{(k)}, \end{cases}$$

$$\delta^{(mk)} = \begin{cases} 0 & \text{if } m \neq k, \\ 1 & \text{if } m = k, \end{cases}$$

$$z^{(k)} = \begin{cases} \overline{p}^{(k)} \text{ if } \phi \text{ is specified over } C^{(k)}, \\ \overline{\phi}^{(k)} \text{ if } \partial\phi/\partial n \text{ is specified over } C^{(k)}. \end{cases} \qquad (1.31)$$

Note that $z^{(1)}$, $z^{(2)}$, $\cdots$, $z^{(N-1)}$ and $z^{(N)}$ are the N unknown constants on the right hand side of Eq. (1.27), while $a^{(mk)}$ and $b^{(mk)}$ are known coefficients.

Once Eq. (1.30) is solved for the unknowns $z^{(1)}$, $z^{(2)}$, $\cdots$, $z^{(N-1)}$ and $z^{(N)}$, the values of ϕ and $\partial\phi/\partial n$ over the element $C^{(k)}$, as given by $\overline{\phi}^{(k)}$ and $\overline{p}^{(k)}$ respectively, are known for $k = 1$, 2, $\cdots$, N. Eq. (1.27) with $\lambda(\xi, \eta) = 1$ then provides us with an explicit formula for computing ϕ in the interior of R, that is,

$$\phi(\xi, \eta) \simeq \sum_{k=1}^{N} \{\overline{\phi}^{(k)} \mathcal{F}_2^{(k)}(\xi, \eta) - \overline{p}^{(k)} \mathcal{F}_1^{(k)}(\xi, \eta)\} \text{ for } (\xi, \eta) \in R. \tag{1.32}$$

To summarize, a boundary element solution of the interior boundary value problem defined by Eqs. (1.1)-(1.2) is given by Eq. (1.32) together with Eqs. (1.28), (1.30) and (1.31). Because of the approximation in Eqs. (1.25) and (1.26), the solution is said to be obtained using constant elements. Analytical formulae for calculating $\mathcal{F}_1^{(k)}(\xi, \eta)$ and $\mathcal{F}_2^{(k)}(\xi, \eta)$ in Eq. (1.28) are given in Eqs. (1.37), (1.38), (1.40) and (1.41) (together with Eq. (1.35)) in the section below.

1.6 Formulae for Integrals of Constant Elements

The boundary element solution above requires the evaluation of $\mathcal{F}_1^{(k)}(\xi, \eta)$ and $\mathcal{F}_2^{(k)}(\xi, \eta)$. These functions are defined in terms of line integrals over $C^{(k)}$ as given in Eq. (1.28). The line integrals can be worked out analytically as follows.

Points on the element $C^{(k)}$ may be described using the parametric equations

$$\left.\begin{array}{l} x = x^{(k)} - t\ell^{(k)} n_y^{(k)} \\ y = y^{(k)} + t\ell^{(k)} n_x^{(k)} \end{array}\right\} \text{ from } t = 0 \text{ to } t = 1, \tag{1.33}$$

where $\ell^{(k)}$ is the length of $C^{(k)}$ and $[n_x^{(k)}, n_y^{(k)}] = [y^{(k+1)} - y^{(k)}, x^{(k)} - x^{(k+1)}]/\ell^{(k)}$ is the unit normal vector to $C^{(k)}$ pointing away from R.

For $(x,y) \in C^{(k)}$, we find that $ds(x,y) = \sqrt{(dx)^2 + (dy)^2} = \ell^{(k)} dt$ and

$$(x-\xi)^2 + (y-\eta)^2 = A^{(k)} t^2 + B^{(k)}(\xi,\eta) t + E^{(k)}(\xi,\eta), \quad (1.34)$$

where

$$\begin{aligned} A^{(k)} &= [\ell^{(k)}]^2, \\ B^{(k)}(\xi,\eta) &= [-n_y^{(k)}(x^{(k)} - \xi) + (y^{(k)} - \eta) n_x^{(k)}](2\ell^{(k)}), \\ E^{(k)}(\xi,\eta) &= (x^{(k)} - \xi)^2 + (y^{(k)} - \eta)^2. \end{aligned} \quad (1.35)$$

For any point (ξ,η), the parameters in Eq. (1.35) satisfy $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 \geq 0$. To see why this is true, consider the straight line defined by the parametric equations $x = x^{(k)} - t\ell^{(k)} n_y^{(k)}$ and $y = y^{(k)} + t\ell^{(k)} n_x^{(k)}$ for $-\infty < t < \infty$. Note that $C^{(k)}$ is a subset of this straight line (given by the parametric equations from $t = 0$ to $t = 1$). Eq. (1.34) also holds for any point (x,y) lying on the extended line. If (ξ,η) does not lie on the line then $A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta) > 0$ for all real values of t (that is, for all points (x,y) on the line) and hence $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 > 0$. On the other hand, if (ξ,η) is on the line, we can find exactly one point (x,y) such that $A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta) = 0$. As each point (x,y) on the line is given by a unique value of t, we conclude that $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 = 0$ for (ξ,η) lying on the line.

From Eqs. (1.28), (1.33) and (1.34), $\mathcal{F}_1^{(k)}(\xi,\eta)$ and $\mathcal{F}_2^{(k)}(\xi,\eta)$ may be written as

$$\begin{aligned} \mathcal{F}_1^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{4\pi} \int_0^1 \ln[A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta)]dt, \\ \mathcal{F}_2^{(k)}(\xi,\eta) &= \frac{\ell^{(k)}}{2\pi} \int_0^1 \frac{n_x^{(k)}(x^{(k)} - \xi) + n_y^{(k)}(y^{(k)} - \eta)}{A^{(k)}t^2 + B^{(k)}(\xi,\eta)t + E^{(k)}(\xi,\eta)} dt. \end{aligned}$$

(1.36)

The second integral in Eq. (1.36) is the easiest one to work out for the case in which $4A^{(k)}E^{(k)}(\xi,\eta) - [B^{(k)}(\xi,\eta)]^2 = 0$. For