Radiative Transfer Using Boltzmann Transport Theory

by

Carnell Littlejohn

ISBN: 1-58112- 256-X

DISSERTATION.COM

Boca Raton, Florida
USA • 2004

Radiative Transfer Using Boltzmann Transport Theory

Dissertation.com
Boca Raton, Florida
USA • 2004

ISBN: 1-58112- 256-X

Radiative Transfer Using Boltzmann Transport Theory

by

Carnell Littlejohn
Department of Mathematics and Computer Science
Chicago State University
October 1997
Revised August 1998

In Partial Fulfillment of the Requirements for the Master of Science Degree in Mathematics

Advisor: Dr. Richard Solakiewicz

Title : Radiative transfer using Boltzmann transport theory

Abstract:

Radiative transfer of photons though a random distribution of scatterers is considered. The Boltzmann transport eq is used to develop a program to obtain real values of intensity based on a set of discrete time intervals. A Newton-Raphson method is used to determine a set of eigenvalues based on the boundary conditions and system geometry. A numerical method due to Lanczos is used to approximately invert a Laplace transform. The algorithm is designed for easy modification to more general problems.
Please take note that a 3.5 inch disk for eigenvalue and intensity calculations is available with the revised thesis.

PART II

TITLE: LEGENDRE POLYNOMIALS AND EIGENVALUES

Abstract:
SEE REFERENCE [1] : Radiative transfer using Boltzmann transport theory

(1) CHICAGO STATE UNIVERSITY LIBRARY Call Number:Z5053.L588 c.2
(2) UNIVERSITY OF CHICAGO LIBRARY , TEXT + 3 ½ in. COMPUTER DISK
 CRERAR LIB STACKS , C.1 TEXT, Call Number: QC175.25.R3L588 1999
 CRERAR LIB Media Coll . , C.1 ComDisk , Call Number:QC175.25.R3L588 1999
(3) MASTERS ABSTRACT INTERNATIONAL VOL 37\01\ 1999 p.272
(4) FIRST SEARCH \ WORLDCAT : Accession Number(s) : 38920963 ,39034458,4385899

Eigenvalues for the intensity distribution from the one speed Boltzmann transport equation can be computed using an iteration method with the roots and coefficients for the zeros of the Legendre polynomials and the results converge to the known values based on a Newton-Raphson method used in an earlier treatment for a radiative transfer . A spherical harmonic expansion of the intensity and the application of laplace and finite fourier transforms was used to solve the problem . The zero of a transcendental equation based on the differential equation for the boundary conditions was approximated for the eigenvalues . A Newton-Raphson method was used . Here an iteration method is used with the Legendre polynomials and the results are identical to the Newton-Raphson results therefore the Legendre polynomials can be used to compute the eigenvalues for the intensity distribution .
A S16 approximation of the intensity distribution using Fast Fourier Transforms for the coefficients of the interpolating polynomial is given . The program for the coefficients is from a standard Numerical Methods text (Burden and Faires, 1993, Chap.8, p. 309) . Arctans from the approximation are compared with the Newton-Raphson eigenvalues .

The Thesis was presented for the Mathematics 495 requirement at Chicago State University in October 1997. In December 1997 the abstract for (Su and Olson, September 1997) was obtained from the INSPEC Database at The University of Chicago John Crerar Library. The advisor for the Thesis; Dr. Solakiewicz; obtained a copy of the article in March 1998. Due to the similar treatments for The Thesis and (Su and Olson September 1997) of the transport equation; mainly the application of the Laplace and Fourier transforms for the solutions in addition to numerical inversion codes; Pascal in the case of the Thesis and Fortran in the case of (Su and Olson, September 1997); it was decided to include a section on the results presented in the *Annals of Nuclear Energy* by Su and Olson and a comparison with plots used for the presentation of the Thesis. It is also noted that in both cases a Diffusion approximation was used for comparison. The Thesis used (Koshak: et al 1994) where a Monte-Carlo approach was simulated with a mean free path,polar and azimuthal angle setting,vertical optical depth and single scattering albedo for a cubical cloud. Su and Olson used (Su and Olson,1996) where the above mentioned Fortran codes were used for the numerical evaluations of the expressions for the diffusion approximation which is also included in the *Annals of Nuclear Energy*.

Also included in this revision of the Thesis is a standalone routine for eigenvalue calculations. The routine is concerned with eigenvalues for any s and error bound or tolerance. The transcendental equation is evaluated at the eigenvalue and the magnitude of the result is considered the error itself. This ideally should be zero. The results obtained are of the order of 10 to minus 10th. The Newton-Raphson method is the basis for the calculations with 3 to 5 iterations . After the 3rd iteration or so the difference between successive values go to zero. The Gaussian method is also used with Legendre polynomials of degree 2 ,3,5,6,7,9,10,11,12,13,15,17,18,19,20,21 and 25.The roots and coefficients are used in the polynomials with the roots as the variable or arguments for the zero of the polynomial. The coefficients and roots are taken from (Secrest and Stroud, 1966).

The evaluation of the trancendental equation with the roots and coefficients produces the eigenvalue. For hcot(ax)-x=0 or cot(ax) = x/h; cot(ax) with roots and coefficients or x/h with roots and coefficients can be used. For certain values of s the agreement between the Newton and Gaussian methods are good. In some cases it is necessary to multiply the Gaussian result by a positive integer to obtain the value or a value near the Newton-Raphson result.

A DOS based Turbo Pascal program is used. The eigenvalue calculations are done in a unit with object code linked into the main Intensity calculations where the eigenvalue information for Intensity is input at the DOS prompt as well as the Legendre polynomial information for the inversion of the Laplace transform.

For the EXE form of the DOS based program the disk is inserted into the A drive

A:\> FILENAME

FILENAME is the name of the DOS based program with the exe extension. The name of the program is typed without the exe extension. The DOS version should be 6.0 or higher. For WINDOWS a DOS PIF is needed to run the program and the PIF information may or maynot be available with the DOS prompt. Once the filename is typed this or a similar message appears on the screen

A:\> FILENAME

**This is a Pascal unit for eigenvalue calculations.
The Newton-Raphson and Gaussian Quadrature methods from the Radiative Transfer using Boltzmann Transport Theory are used.**

**THE METHOD USED HERE FOR THE LEGENDRE POLYNOMIALS IS AN ADDITION METHOD
SEE APPENDIX TABLE 4 FOR DETAILS**

After the message appears the user prompts allow parts of or all of the routines to execute. For the data list of Time and Intensity the user is prompted to exit and type the command

A:\>TYPE FILENAME.TEX or A:\> Fileview for Fileview.EXE and use OPEN to view contents of file.

This is a sample text file output from text file with name FILENAME.The extension for this text file is 'tex' and the extension is used to view the contents of the file.

TIME	INTENSITY
0.0000000000E+00	0.0000000000E+00
1.0000000000E+00	2.8790048608E+01
2.0000000000E+00	5.7256208535E+00
3.0000000000E+00	2.2365365307E+00
4.0000000000E+00	1.3973724815E+00
5.0000000000E+00	1.1388879043E+00
TIME	INTENSITY

VALUES FOR TIME AND INTENSITY FOR 5.0000000000E+00 MICROSECONDS

The standalone eigenvalue results are directed to a text file. The text file is called when the standalone eigenvalue results are requested. At this time code is being developed for the use of Object oriented, unit oriented Turbo Pascal for Windows applications directed towards the computations for future treatments of Radiative transfer.

Carnell Littlejohn
March 1998

TABLE OF CONTENTS (revised)

Radiative Transfer Using Boltzmann Transport Theory

by

Carnell Littlejohn

Radiative transfer of photons though a random distribution of scatterers is considered. The Boltzmann transport eq is used to develop a program to obtain real values of intensity based on a set of discrete time intervals. A Newton-Raphson method is used to determine a set of eigenvalues based on the boundary conditions and system geometry. A numerical method due to Lanczos is used to approximately invert a Laplace transform. The alogrithm is designed for easy modification to more general problems.

Introduction

Radiative transfer of photons though a random distribution of scatterers is considered. This is a model for the propagation of light due to lightning though a thundercloud. Photons random walk though a fixed lattice of cloud droplets. In order to make the problem tractable, the cloud is modeled as a right parallepiped containing only identical spherical scatterers. We assume that each scatter has a radius of 10 μm and their number density is $100\ cm^{-3}$. Physical properties are taken to apply at a wavelegnth of $\lambda = 0.7774 \mu m$ (near infrared).The source will be taken to be an impulse (a delta function) located at the cloud's center. Photon-photon interactions and effects such as gravitational redshifts are neglected.

Several methods have been employed to study this problem: Monte-Carlo [1], Equivalent Medium [2] and Diffusion Approximation [3]. In the Monte-Carlo method, a large number of photons are tracked as they travel through the cloud medium. Their histories are dictated by the generation of random numbers together with a knowledge of probabilities for absorption and scattering.The collection of drops is considered as a single synthethic medium in the Equivalent Medium approach. Classical methods of electromagnetics are then applied. The method used here is most similar to the Diffusion Approximation, but the boundary conditions are treated more precisely.

This work begins with the Boltzmann eq. written for photon intensity. The intensity depends on position, velocity direction and time. A 5-dimensional phase-space is employed. In order to reduce the number of dimensions, we work with an expansion of the intensity in spherical harmonics.This leads to a hieriarchy of coupled differential eqs. The hieriarchy is truncated using the Eddington approximation [4].The coeficients of the spherical harmonics depend on space and time. A Laplace transform followed by a set of finite Fourier transforms are employed to solve the problem.The Fourier transforms may be inverted analytically. A numerical method [5] is used to invert the Laplace transform. Data was generated using a computer. A graph of intensity vs. time is displayed and compared with similar graphs in the existing literature.

The Problem

We are attempting to model radiative transfer in a thundercloud. The cloud will be assumed to be composed of a random distribution of spherical water droplets with a common radius of 10 μm. The number density is taken to be $\rho = 100\ \text{cm}^{-3}$. Photons produced by lightning can interact with the water droplets in several ways. If conservative scattering (no change in wavelegnth) is assumed, the interaction is either an absorbtion of the photon by the water droplet or it is a scattering event.The probability that a collision between a photon and a water droplet results in a scattering rather than an absorbtion is $\omega_0 = 0.99996$ (known as the single scattering albedo). The probability that a photon scatters in a particular direction $\hat{\Omega}$ given an incidence along $\hat{\Omega}'$ is given by a phase function $\wp(\hat{\Omega},\hat{\Omega}')$.

The intensity (in watts per square meter per steradian) is related to the number density in phase space and it satisfies

$$\frac{\partial I}{\partial t} = -c\hat{\Omega}\cdot\nabla I + \frac{c}{4\pi}K\omega_0\int \wp I d\hat{\Omega}' - KcI + cS. \tag{1}$$

Here, c is the constant speed of light, K is the inverse mean free path and S is a source term.The time rate of change in photon intensity is given as the sum of 4 terms. The advection term $-c\hat{\Omega}\cdot\nabla I$ may be recognized as part of what is known as a material derivative

$$\frac{DI}{Dt} = \frac{\partial I}{\partial t} + (\mathbf{V}\cdot\nabla)I,$$

where the velocity $\mathbf{V} = c\hat{\Omega}'$ has a constant magnitude (speed) in this case.The next term on the right side of (1) is known as the collision integral. It represents the associated intensity for the number of photons in a given volume traveling along any direction $\hat{\Omega}'$ that are scattered into $\hat{\Omega}$. Subtracted from this is the intensity associated with photons which are scattered out of the volume.The source term represents photon production.The phase function we will be using is given by

$$\wp(\hat{\Omega},\hat{\Omega}') = \wp(\hat{\Omega}\cdot\hat{\Omega}')\frac{1-g^2}{(g^2-2g\varsigma+1)^{3/2}},\ \varsigma = \hat{\Omega}\cdot\hat{\Omega}'; \tag{2}$$

$g = 0.84$ is the asymetry factor. This is known as the Henyey-Greenstein function [6].

The condition at an interface between a convex cloud and vacaum is given by

$$I(\mathbf{r},\ \hat{\Omega},\ t) = 0,\ \hat{\Omega}\cdot\hat{\mathbf{n}} > 0, \tag{3}$$

where $\mathbf{r}$ is a position vector on a cloud's boundary and $\hat{\mathbf{n}}$ is the outward normal at $\mathbf{r}$.Once a photon leaves a convex cloud there is no mechanism for it to reenter.There are no scatterers to change the photon's direction. At a time before any lightning flash occurs, $I = 0$ everywhere. This provides the intial condition for the problem.

Mathematical Method

Eq 1 is written in a 5- dimensional phase space. It consists of 3 dimensions for position and 2 dimensions (e.g. the usual θ and ϕ in a spherical coordinate system) for velocity direction. Similar problems in neutron reactor theory use a 6- dimensional phase space since the speed of the particles in question is not constant. Reduction to 3-dimensions may be acomplished by expansion in spherical harmonics. The intensity may be written as

$$I(\mathbf{r},\hat{\Omega},t) = b_0^{\,0}(\mathbf{r},t) + b_1^{\,-1}(\mathbf{r},t)Y_1^{\,-1}(\hat{\Omega}) + b_1^{\,0}(\mathbf{r},t)Y_1^{\,0}(\hat{\Omega}) + b_1^{\,1}(\mathbf{r},t)Y_1^{\,1}(\hat{\Omega}) + \ldots \tag{4}$$

Spherical harmonics are defined by [7]

$$Y_n^{\,m}(\hat{\Omega}) = P_n^{\,m}(cos\theta)e^{im\varphi},$$
$$P_n^{\,m}(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n, \quad P_n^{\,-m}(x) = \frac{(-1)^m(n-m)!}{(n+m)!}P_n^{\,m}(x). \tag{5}$$

These functions satisfy the orthogonality relation

$$\int_{4\pi} d\Omega Y_\nu^{\,-\mu}(\hat{\Omega})Y_n^{\,m}(\hat{\Omega}) = (-1)^m\frac{4\pi}{2n+1}\delta_{\mu m}\delta_{\nu n}, \tag{6}$$

where δ_{mn} is the Kronecker delta. The coeficients in such an expansion of the Henyey Greenstein function are $d_n^{\,m} = (2n+1)g^n\delta_{m0}$. The advection term may be written using

$$\hat{\Omega}\cdot\nabla \equiv sin\theta cos\varphi\frac{\partial}{\partial x} + sin\theta sin\varphi\frac{\partial}{\partial y} + cos\theta\frac{\partial}{\partial z}. \tag{7}$$

Absorbing the trigonometric function into the spherical harmonics using recursion formulas [8] and using orthogonality results in an infinite set of coupled differential eqs. See [9] for details.

Setting $b_0^{\,0} = f_1$, $b_1^{\,1} - \dfrac{b_1^{\,-1}}{2} = f_2$, $i\left(b_1^{\,1} + \dfrac{b_1^{\,-1}}{2}\right) = f_3$, and $b_1^{\,0} = f_4$, we may write

$$\frac{1}{c}\frac{\partial f_1}{\partial t} + \beta_0 K f_1 = -\frac{1}{3}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y} + \frac{\partial f_4}{\partial z}\right) + S_0^{\,0};$$
$$\frac{1}{c}\frac{\partial f_2}{\partial t} + \beta_1 K f_2 = -\frac{\partial f_1}{\partial x},\ \frac{1}{c}\frac{\partial f_3}{\partial t} + \beta_1 K f_3 = -\frac{\partial f_1}{\partial y},\ \frac{1}{c}\frac{\partial f_4}{\partial t} + \beta_1 K f_4 = -\frac{\partial f_1}{\partial z}, \tag{8}$$
$$\beta_n = 1 - \frac{\omega_0 d_n}{2n+1}.$$

We assume that the source is isotropic; S_0^0 is therefore the only non-zero coeficient in the expansion of the source. Futhermore the hierachy of eqs. generated was truncated at $n = 1$.This is the essence of the Eddington approximation. Rationale for the plausability of this approximation provided in [3].

Direct application of the boundary condition is in general not possible once the Eddington approximation is made. This condition is customarily replaced by the lowest order Marshak boundary condition [8].

$$F_n = \int_{\hat{\Omega}\cdot\hat{\mathbf{n}} < 0} \hat{\Omega}\cdot\hat{\mathbf{n}} I(\mathbf{r},\hat{\Omega},t) d\Omega = 0, \tag{9}$$

where F_n is the irradiance in the direction $\hat{\mathbf{n}}$. The integral of F_n over a portion of the interface yields the number of photons reentering the cloud though that portion.

Consider a plane surface separating the cloud and the region outside with normal $\hat{\mathbf{n}}$.Since we may take the area to be vanishingly small, (9) may be rewritten as

$$\int_0^{2\pi} d\phi_n \int_{\frac{\pi}{2}}^{\pi} d\theta_n sin\theta_n cos\theta_n I[\theta_n,\phi_n] = 0, \tag{10}$$

where $I[\theta_n,\phi_n]$ is evaluated on the cloud boundary and θ_n is measured from $\hat{\mathbf{n}}$ considered as a polar axis. An appropriate rotation of cordinates yields the condition

$$f_1 = \frac{2}{3}(n_x f_2 + n_y f_3 + n_z f_4) \quad \hat{\mathbf{n}} = n_x \hat{\mathbf{x}} + n_y \hat{\mathbf{y}} + n_z \hat{\mathbf{z}} \tag{11}$$

on the interface. In [3], a diffusion approximation was used to put the problem into a standard Strum-Liouville form which has an analytic solution. Here, we follow [10] and retain the Marshak boundary condition. Further progress is made by employing transform methods.

We begin solving (8) by taking Laplace transforms. The Laplace transform pair is given by [11]

$$F(\mathbf{r},s) = \int_0^{\infty} e^{-st} f(\mathbf{r},t) dt, \quad f(\mathbf{r},t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} F(\mathbf{r},s) ds, \tag{12}$$

where all of the poles of $F(\mathbf{r},s)$ have real parts greater than α. We will be using a numerical method for inverting the Laplace transform rather than the complex inversion formula.

Operationally we have

$$\int_0^\infty e^{-st}\frac{\partial}{\partial t}f(\mathbf{r},t) = sF(\mathbf{r},s) - f(\mathbf{r},0). \tag{13}$$

Numerous other properties are available in the literature; only one time derivative is present in this problem. Employing Laplace transforms allows us to rewrite (8) as

$$\begin{gathered}\left(\beta_0 K + \frac{s}{c}\right)F_1 = \frac{1}{3}\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial y} + \frac{\partial F_4}{\partial z}\right) + S + \frac{f_1(0)}{c}\\ \left(\beta_1 K + \frac{s}{c}\right)F_2 = -\frac{\partial F_1}{\partial x},\ \left(\beta_1 K + \frac{s}{c}\right)F_3 = -\frac{\partial F_1}{\partial y},\ \left(\beta_1 K + \frac{s}{c}\right)F_4 = -\frac{\partial F_1}{\partial z},\\ F(\mathbf{r},s) = \int_0^\infty e^{-st}f(\mathbf{r},t)dt,\ S(\mathbf{r},s) = \int_0^\infty e^{-st}S(\mathbf{r},t)dt.\end{gathered} \tag{14}$$

Substituting the second, third and fourth eqs. into the first gives

$$\begin{aligned} k^2F_1 &= \nabla^2 F_1 + T,\\ k^2 &= 3\left(\beta_0 K + \frac{s}{c}\right)\left(\beta_1 K + \frac{s}{c}\right),\\ T &= 3\left(\beta_1 K + \frac{s}{c}\right)\left(S + \frac{f_1(0)}{c}\right),\\ \nabla^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.\end{aligned} \tag{15}$$

The boundary condition (11) takes the form

$$\begin{aligned} \frac{\partial F_1}{\partial n} + hF_1 = 0,\ h &= \frac{3\left(\beta_1 K + \frac{s}{c}\right)}{2},\\ \frac{\partial F_1}{\partial n} &= n_x\frac{\partial F_1}{\partial x} + n_y\frac{\partial F_1}{\partial y} + n_z\frac{\partial F_1}{\partial z}.\end{aligned} \tag{16}$$

Eqs. (15) and (16) have the form of a diffusion problem with Robin-type boundary conditions.The solution to (15) and (16) may be obtained using finite Fourier transforms [12]. A transform pair is selected so that the boundary conditions will be satisfied.

For a function $u(x)$, a transform pair can be defined by

$$U(n) = \int_{-\frac{A}{2}}^{\frac{A}{2}} u(x)cos\zeta_n x dx \text{ and } u(x) = \sum_{n=1}^{\infty} A_n cos\zeta_n x; \tag{17}$$

ζ_n and A_n are to be determined. For a Robin-type boundary condition

$$\left(\frac{\partial u}{\partial x} + hu\right)\Big|_{x=A} = \sum_{n=1}^{\infty} A_n\left[-\zeta_n sin\zeta_n \frac{A}{2} + hcos\zeta_n \frac{A}{2}\right] = 0, \tag{18}$$

where $\frac{\partial}{\partial n} = \frac{\partial}{\partial x}$ at $x = \frac{A}{2}$. The boundary condition will be satisfied if ζ_n is chosen according to

$$tan\zeta_n \frac{A}{2} + \frac{h}{\zeta_n}. \tag{19}$$

The values of A_n are obtained using orthogonality. The transform pair

$$U(n) = \int_{-\frac{A}{2}}^{\frac{A}{2}} f(x)cos\zeta_n x dx \text{ and } u(x) = \sum_{n=1}^{\infty} \frac{2h}{Ah + sin^2\zeta_n \frac{A}{2}} U(n)cos\zeta_n x; \quad tan\zeta_n \frac{A}{2} = \frac{h}{\zeta_n} \tag{20}$$

is obtained this way.

Operationally, we have

$$\int_{-\frac{A}{2}}^{\frac{A}{2}} u''(x)cos\zeta_n x = -\zeta^2 \int_{-\frac{A}{2}}^{\frac{A}{2}} u(x)cos\zeta_n x dx. \tag{21}$$

This transformation can turn the coupled system of differential equations into an algebraic system.

Applying (21) in all 3 dimensions gives

$$F_1 = \int_{-\frac{A}{2}}^{\frac{A}{2}} dx cos\zeta_p x \int_{-\frac{B}{2}}^{\frac{B}{2}} dy cos\eta_q y \int_{-\frac{C}{2}}^{\frac{C}{2}} dz cos\nu_r z F_1 \tag{22}$$

for a cloud with dimensions A,B,C. Eq(15) may be written as

$$k^2 F_1 = -(\zeta_p^2+\eta_q^2+\nu_r^2)F_1+T_0,$$

$$T_0 = \int_{-\frac{A}{2}}^{\frac{A}{2}} dx cos\zeta_p x \int_{-\frac{B}{2}}^{\frac{B}{2}} dy cos\eta_q y \int_{-\frac{C}{2}}^{\frac{C}{2}} dz cos\nu_r z T. \tag{23}$$

The transformed solution is simply

$$F_1 = \frac{T}{\zeta^2+\eta^2+\nu^2+k^2}. \tag{24}$$

The transformed solution was obtained by first taking a Laplace transform and then finite Fourier transforms.The reverse transformation must be done in reverse order. First we take the inverse finite Fourier transform of (24) and follow that by finding the inverse Laplace transform. Applying the inversion formula in (20) to (24) gives

$$F_1 = \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{r=1}^{\infty} \frac{8h^3 cosx\zeta_p cosy\eta_q coszv_r F_1}{\left(Ah+\sin^2\zeta_p\frac{A}{2}\right)\left(Bh+\sin^2\eta_q\frac{B}{2}\right)\left(Ch+\sin^2\nu_r\frac{C}{2}\right)}. \tag{25}$$

The remaining Laplace transformed coeficients may be recovered from (14). Since trancendental eqs. must be solved to obtain the eigenvalues (ζ_p, η_q, ν_r) and these depend on s it is unlikely that an analytic inverse can be found using the complex inversion formula. Instead, a numerical procedure is employed which requires values of the Laplace transform of a function evaluated at the positive integers.

The basis for this numerical method is to define a set of shifted Legendre polynomials which are orthogonal on $[0,1]$ rather than on $[-1,1]$. This is accomplished by 2 substitutions. In the Laplace transform, we make the change of variable $t = -\ln[s]$ to obtain

$$F(p) = \int_0^\infty e^{-pt} f(t)dt = -\int_{-1}^{1} s^{p-1} f(-\ln[s])ds = \int_0^1 f(s)s^{p-1}ds, \tag{26}$$

where $f(s) = f(-\ln[s]) = f(t) = f[e^{-t}]$. Note that the limits of integration are 0 and 1.

Legendre poloynomials satisfy

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{m,n}. \tag{27}$$

If we make the substitution $y = \frac{x+1}{2}$ it is found that

$$\begin{aligned} \int_{-1}^{1} P_m(x)P_n(x)dx &= 2\int_{-1}^{1} P_m(2y-1)dyP_n(2y-1)dy = \frac{2}{2n+1}\delta_{m,n} \\ \int_0^1 P_m^*(y)P_n^*(y) &= \frac{\delta_{m,n}}{2n+1}, \end{aligned} \tag{28}$$

where $P_n(2y-1) = P_n^*(y)$.These are known as shifted Legendre polynomials.Their orthogonality properties are similar to those of the Legendre polynomials only that the interval is $[0,1]$ rather than $[-1,1]$.The interval for applying orthogonality for the shifted Legendre polynomials is the same as for the Laplace transform as formulated in (26). A table of the coeficients for the shifted Laplace transform is available [5].These coeficients are all integers.

In the following, it will be convenient to write the shifted Legendre polynomials as

$$P_n^*(x) = \sum_{M=0}^{n} P_n^{*(m)} x^m. \tag{29}$$

Here, $P_n^{*(m)}$ is the coeficient of x^m in the nth degree shifted Legendre polynomial $P_n^{*(m)}$. Starting with the Laplace transform as given in (26) we have

$$F(m+1) = \int_0^1 f[s]s^m ds \tag{30}$$

at the integer $p = m+1$. Multiplying both sides by $P_n^{*(m)}$ and summing over m yields

$$F(m+1)P_n^{*(m)} = \int_0^1 f[s']P_n^{*(m)} s'^m ds',$$

$$\sum_{m=0}^{nF} (m+1)P_n^{*(m)} = \sum_{m=0}^{n} \int_0^1 f(s')P_n^{*(m)} s'^m ds' = \int_0^1 f(s')\sum_{m=0}^{n} P_n^{*(m)} s'^m ds. \tag{31}$$

Recognizing the last sum using (29) shows that

$$\sum_{m=0}^{n} F(m+1)P_n^{*(m)} = \int_0^1 f[s']P_n^{*}s' ds'. \tag{32}$$

Except for a normalizing constant the integral on the right provides a method for obtaining the coeficients for the expansion of a function $f[s']$ in shifted Legendre polynomials. At this point orthogonality may be employed. Multiplying both sides of (32) by the product $(2n+1)$ and the nth degree shifted Legendre polynomial and summing gives

$$\sum_{m=0}^{\infty} (2n+1)P_n^{*}(s)\sum_{m=0}^{n} P_n^{*(m)}F(m+1) = \sum_{n=0}^{\infty}\left[(2n+1)\int_0^1 f[s']P_n^{*}(s')ds'\right]P_n^{*}(s) \tag{33}$$

The factor in brackets equals the $(n+1)$st coeficient of the expansion of $f(s)$ in the shifted Legendre polynomials. It is found that

$$\sum_{n=0}^{\infty} (2n+1)\left[\sum_{m=0}^{n} P_n^{*(m)}F(m+1)\right]P_n^{*}(s) = f[s] = f[e^{-t}] = f(t),$$
$$f(t) = \sum_{m=0}^{\infty} (2n+1)\left[\sum_{m=0}^{n} P_n^{*(m)}F(m+1)\right]P_n^{*}(e^{-t}). \tag{34}$$

The inverse Laplace Transform or primative $f(t)$ of $F(s)$ may be obtained from a knowldge of F at integer values of S. In practice the sum over n is truncated to a finite value N. Since values of $P_n^{*(m)}$ are available up to $n=15$ in [5]; N will be limited by 15 in this work.This limitation brings up another practical consideration. The units for time that are used must be chosen with care. It is desireable to work with numbers that are the same order of magnitude in order to avoid introducing too much round off error. In (20), numerical solution is most convenient when $(m+1)/c$ is roughly of the same magnitude as $\beta_1 K$ see (16). The transform variable $s = m$ has dimensions time^{-1}. If we take t in seconds $c=3x10^8$m/s and $(m+1)/c << \beta_1 K$ for $m = 0,1,...,15$. Choosing the time unit to be in μs $c=300$m/μs makes $(m+1)/c$ not drastically different from $\beta_1 K$ for values of m we will be using. Other choices of time units are possible, but microseconds are standard units which serve our purpose fairly well. Results in our plot will be given as intensity in watts per squaremeter per steradian vs. time in μs.

Applying (34) to (25) yields

$$f_1 = \sum_{n=0}^{N}(2n+1)\sum_{m=0}^{n} F_1(m+1)P_n^{*(m)}P_n(2e^{-t}-1), \tag{35}$$

where we have returned to using Legendre polynomials. Primatives f_2, f_3 and f_4 of F_2, F_3 and F_4 may be obtained the same way.

In order to obtain more explict results, we define a source at (x_0, y_0, z_0) lighting at t_0 as

$$S_0^0(\mathbf{r},t) = w\delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \tag{36}$$

where w is a weighting factor representing the strength of the flash. The Laplace transform is

$$S(\mathbf{r},s) = w\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\ e^{-st_0}. \tag{37}$$

At $t=0$, before any flash occurs, $f_j = 0$ for all j. Substituting (15) and (37) into (23) with $f_1(0) = 0$ gives

$$T = 3\left(\beta_1 K + \frac{s}{c}\right) we^{-st_0} cos\zeta_p x_0 cos\eta_q y_0 cos\gamma_r z_0. \tag{38}$$

We assume that the source is within the cloud.

Assembling results,

$$f_1(\mathbf{r},t) = \sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}$$
$$\times\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{r=1}^{\infty}\frac{27\left((\frac{m+1}{c}) + \beta_1 K\right)^3 cos\zeta_p x cos\eta_q y cos\gamma_r z}{\left[3(\frac{m+1}{c}+\beta_1 K)\frac{A}{2}+sin^2\zeta_p\frac{A}{2}\right]\left[3(\frac{m+1}{c}+\beta_1 K)\frac{B}{2}+sin^2\eta_q\frac{B}{2}\right]\left[3(\frac{m+1}{c}+\beta_1 K)\frac{C}{2}+sin^2\gamma_r\frac{C}{2}\right]}$$
$$\times\frac{3(\frac{m+1}{c}+\beta_1 K)we^{-(m+1)t_0}cos\zeta_p x_0 cos\eta_q y_0 cos\gamma_r z_0}{3(\frac{m+1}{c}+\beta_0 K)(\frac{m+1}{c}+\beta_1 K)+\zeta_p^2+\eta_q^2+\gamma_r^2}P_n^{*(m)}P_n(2e^{-(t-t_0)}-1) \tag{39}$$

Similar forms may be obtained for f_2 though f_4.

In order to facililitate comparison with results in existing literature, we will select $\hat{\Omega}=\hat{\mathbf{z}}$ set the source at the orgin and take the cloud to be a cube $A=B=C$. If the source activates impulsively at $t=0$ $(t_0=0)$ with unit strength $(w=1)$ and an observer is at (0, 0, z) directly above the source the result is

$$I(\mathbf{r},\hat{z},t) = f_1(\mathbf{r},t) + f_4(\mathbf{r},t) = \sum_{n=0}^{N}(2n+1)\sum_{m=0}^{n}$$

$$\times\sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{r=1}^{m}\frac{27\left((\frac{m+1}{c}) + \beta_1 K\right)^2\left[sin\zeta_r z + 2((m+1)/c+ \beta_1 K)cos\zeta_r z\right]}{2\left[3(\frac{m+1}{c}+\beta_1 K)\frac{A}{2}+sin^2\zeta_p\frac{A}{2}\right]\left[3(\frac{m+1}{c}+\beta_1 K)\frac{A}{2}+sin^2\zeta_q\frac{A}{2}\right]\left[3(\frac{m+1}{c}+\beta_1 K)\frac{A}{2}+sin^2\zeta_r\frac{A}{2}\right]}$$

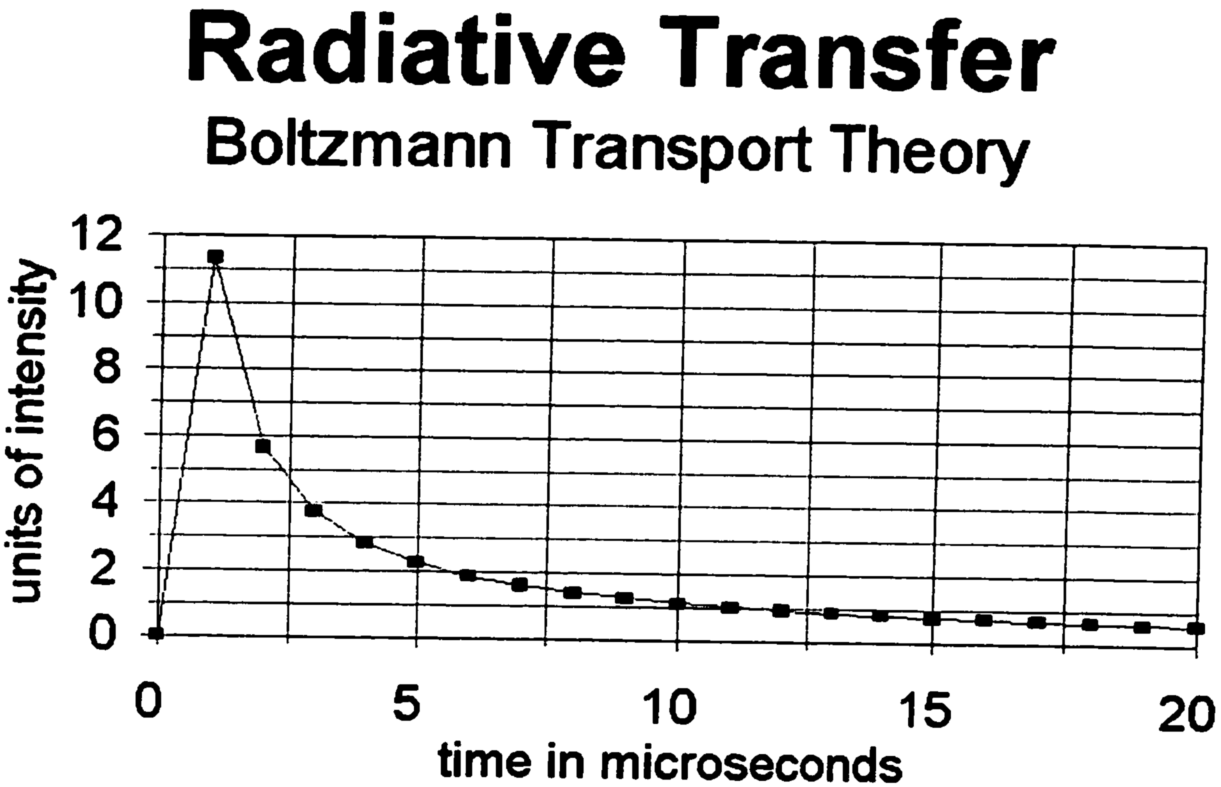
$$\times\frac{3(\frac{m+1}{c}+\beta_1 K)}{3(\frac{m+1}{c}+\beta_0 K)(\frac{m+1}{c}+\beta_1 K)+\zeta_p^2+\zeta_q^2+\zeta_r^2}P_n^{*(m)}P_n(2e^{-t}-1)$$

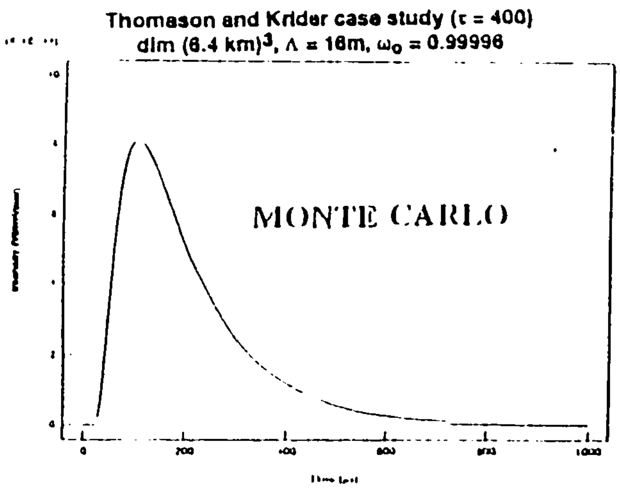
$$tan\zeta_p\frac{A}{2} = \frac{3\left(\frac{m+1}{c} + \beta_1 K\right)}{2\zeta_p}. \qquad (40)$$

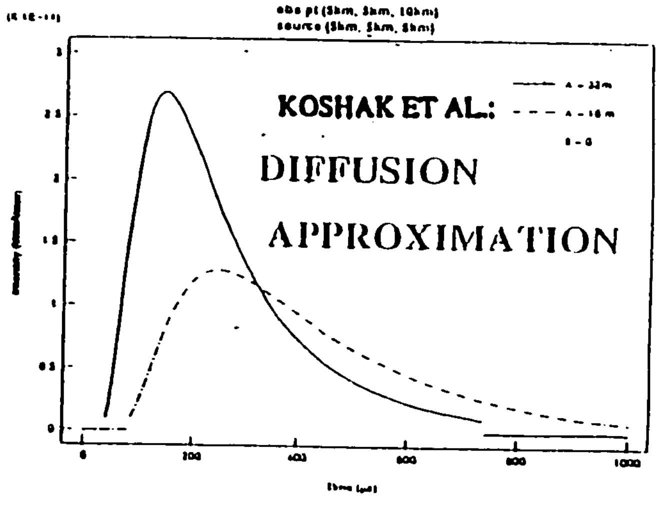
Sufficiently large numbers are to be substituted for m and N. Coeficients of f_2 and f_3 are zero at $\hat{\Omega} = \hat{\mathbf{z}}$.

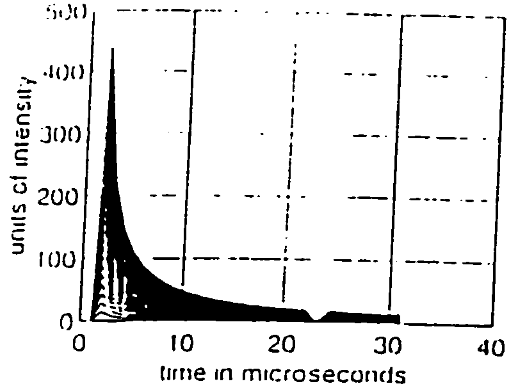
Results

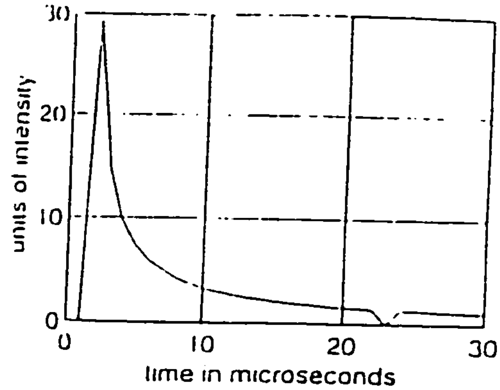
The following plots show intensity in watts per squaremeter per steradian vs time in microseconds for a cubical cloud 10km on a side excited by an impulsive point source at it's center. The observation point is on the top surface of the cloud directly above the source.

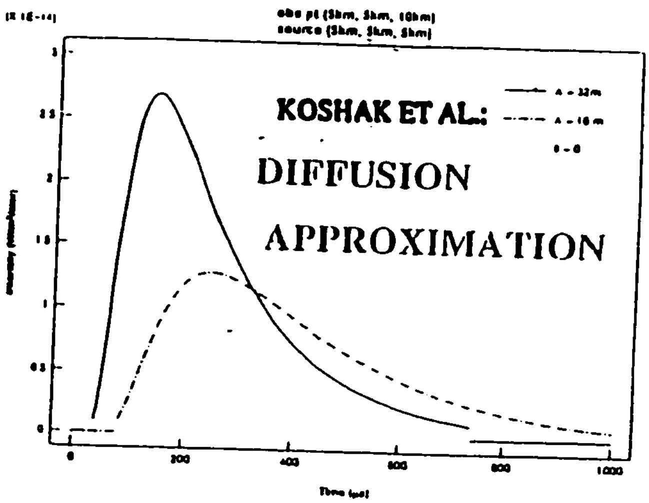








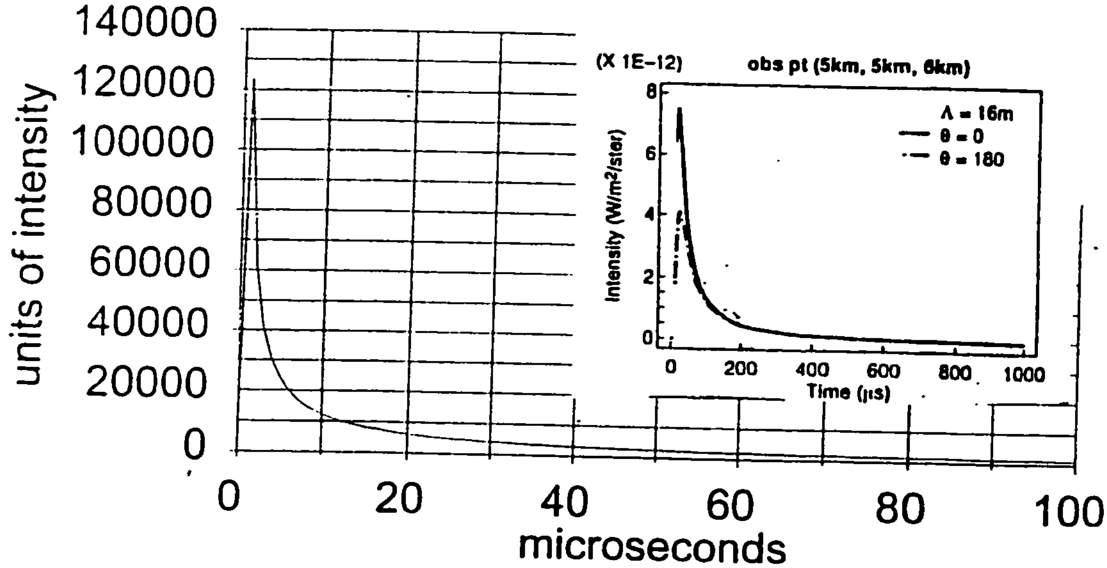




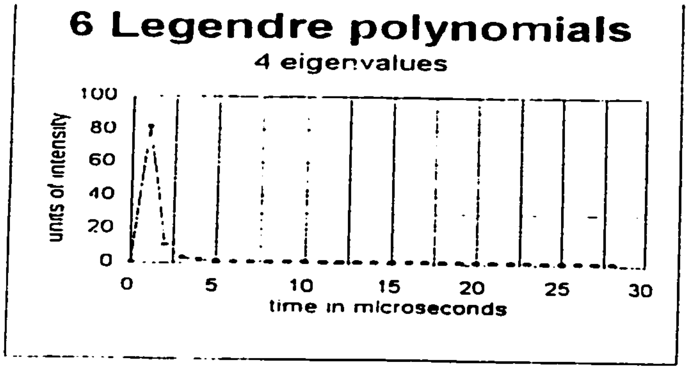
The top figures represent results from the Laplace-Fourier method for radiative transfer. The left figure is a composite of intensity curves. The right figure is a plot of one intensity curve. The positions are set at (x,y,z) = (0km,0km,5km), the location is set at (x,y,z) = (0km,0km,0km). The mean free path Λ is 16m and the angle θ is set at 0.0.The cloud is being looked down upon. The bottom figure is a result from The Journal of Geophysical Research July 1994 Koshak et al.:Lightning Radiative Transfer. The source location is set at (x,y,z) = (5km,5km,5km) and the observation position is set at (x,y,z) = (5km,5km,10km). 'The curves are applicable for any observation altitude over the cloud'.

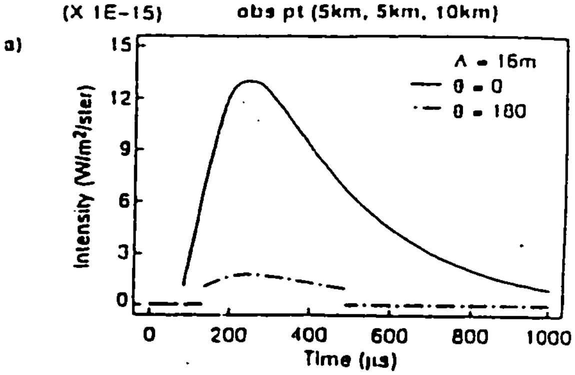
Radiative Transfer Problem

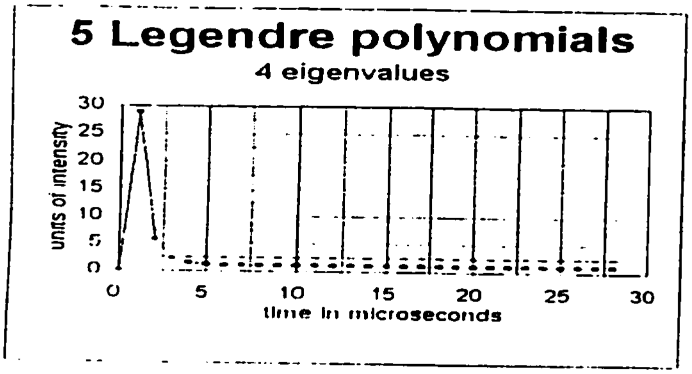
100 Intensity values

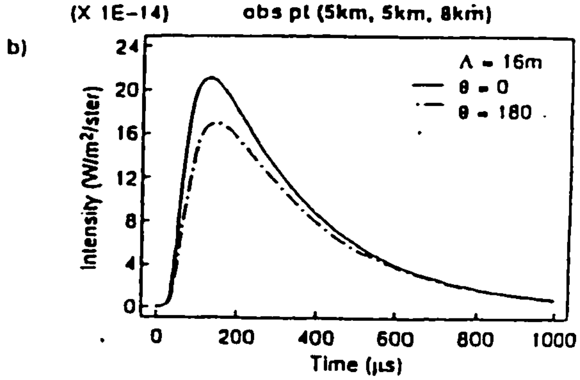


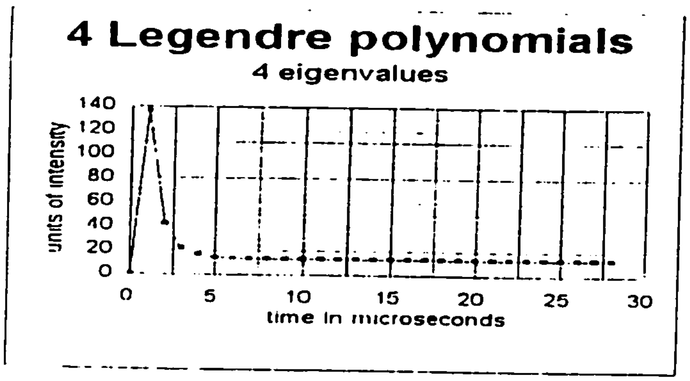
The above is a plot of intensity vs time.The upper right curve is a result from the JOURNAL OF GEOPHYSICAL RESEARCH July 1994 KOSHAK ET.Lightning Radiative Transfer.Waveforms for varying detector locations were considered and the upper right curve is set at Λ = 16m for the mean free path and (x,y,z) = (5km,5km,6km) for the observation point.The angle θ is set at 0.0 and 180.The grided curve is a result from the Laplace-Fourier method.The mean free path Λ is 16m and the angle θ is set at 0.0, the location (x,y,z) is (0km,0km,0km) and the observation point is (x,y,z) = (0km,0km,5km). For the upper right curve and its source location a vertical optical depth of 400 is given as the ratio of the cloud diminsions to the mean free path or 6.4km/16m.The cloud diminsions for the Laplace-Fourier method (x,y,z) is (10km,10km,10km). The sharp peak is explained in terms of closeness to the source.For the upper right curve the sharp peak is clear for θ = 0.0 and 180.The Laplace-Fourier results all contain the sharp peak and the location and observation points are the same for all the curves.For the upper right curve and the grided curve the single scattering albedo is 0.99996.Compare other plots available for Laplace-Fourier results and KOSHAK ET AL. for other detector locations.

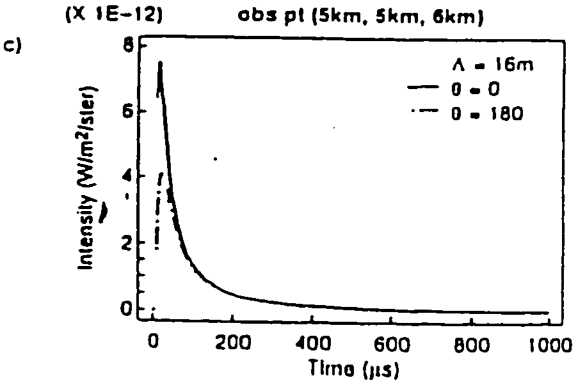










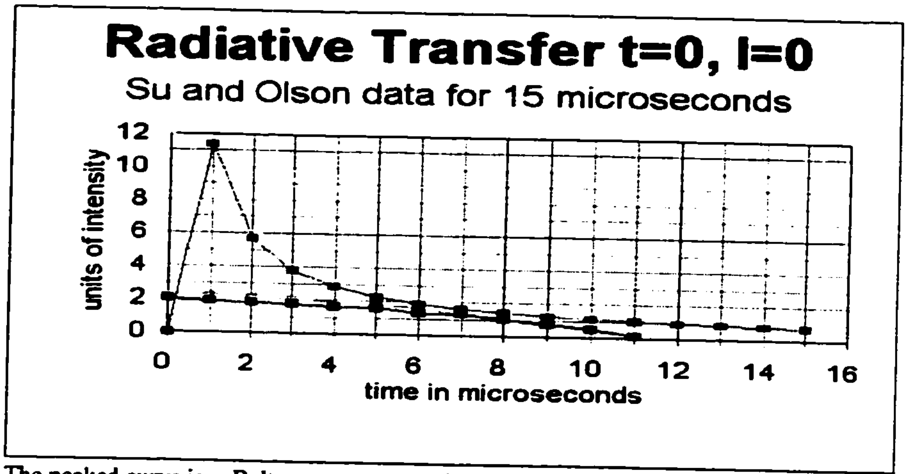


The left column set of plots are results from the Laplace-Fourier method for radiative transfer. The plots are for 28 microseconds and 4 eigenvalues. The source location and observation position are the same for the 3 plots. The number of Legendre polynomials were varied for each plot. The right column is a result from the diffusion approximation [3] where detector locations were varied for the plots.

The plots compare favorably. Using the more precise treatment of boundary conditions does not appear to make much difference. This is expected in light of the estimates provided in [3] regarding the applicability of the diffusion approximation. We do notice a sharper peak and somewhat faster risetime.

Summary

A solution involving numerical inversion of a Laplace transform for a radiative transfer was presented. The results were applied to a cloud modeled as a fixed lattice of identical spherical water droplets. While the droplets comprising the cloud move, their speeds are small enough compared to the speed of light that they may be considered stationary. Intensity was obtained by approximately solving an appropriate form of the Boltzmann transport eq.. This eq. written in 5-dimensional space, was simplified by expanding the intensity in a series of spherical harmonics.The resulting hierarchy of coupled differential equations were solved by truncation and application of Laplace and finite Fourier transforms. Results were given for an impulsive point source located at the center of a cubical cloud. These results compared favorably with those in the existing literature. Further research can be conducted by applying the solution given here to a collection of point sources sequentially lighted to simulate a lightning event. More terms could be retained in the series of Legendre polynomials for inverting the Laplace transform. Finally more terms in the expansion of $I(\mathbf{r},\hat{\Omega},t)$ in spherical harmonics could be used.

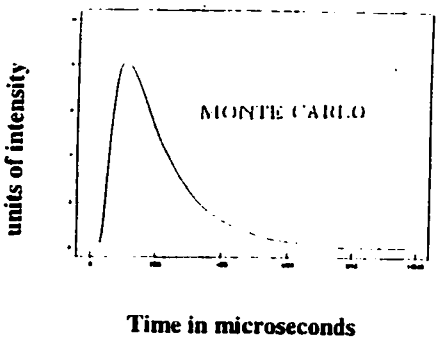


The peaked curve is a Boltzmann transport theory result and the linear type curves near the bottom of the plot are the (Su and Olson, 1997) transport and diffusion results. The Boltzmann plot is intensity in watts per squaremeters per steradians vs time in microseconds while the Su and Olson curves are Radiation energy density vs position for a 'given constant time interval'. The Radiation energy density is defined as the integral of the intensity and the variable of integration is an angular variable on [-1,1].The angular variable for the Boltzmann result generates eigenvalues based on a numerical treatment of a trancendental equation for (pi/2,pi) with a source located at (x,y,z).The unit of time used by Su and Olson is defined in terms of Radiation constants and Heat capacities for a temporal variable.The results should hold for scaled microseconds as the temporal variable.

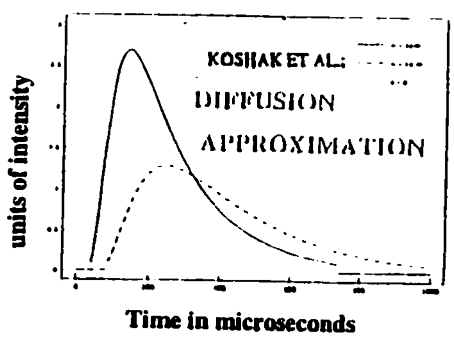
The source is stated as being constant in the time interval; 10 units in this case and existing only for a finite period of time. A unit step function and delta function is used to represent the source. For the Su and Olson data the plots indicate constant nonzero values for earlier times near 0.0 microseconds and decreasing values that tend toward zero in the interval [5,11] microseconds.The Su and Olson transport and diffusion results tend to agree with the Boltzmann transport results in the interval [5,10] microseconds.

A Diffusion Approximaton with Monte-Carlo results for intensity in watts per squaremeters per steradians vs time in microseconds (Koshak et.al:,1994) is compared with the Boltzmann transport theory :Laplace-Fourier results. See plots below. (Su and Olson, 1997) also use a Laplace-Fourier method. See related text and plots for other details.

Thomason and Krider case study ($\tau = 400$)
dim $(6.4km)^3$, $\Lambda = 16m$, $\omega_0 = 0.99996$



obs pt.(5km, 5km,10km)
source(5km, 5km,5km)



(Su and Olson September 1997) presented an analytical transport solution for non-equilibrium **Radiative transfer** in an infinite and isotropically scattering medium.The Radiation energy density is defined in terms of the Intensity and the Hohlraum temperature is taken as a reference temperature. The Material energy density is also defined from the **transport equation** with the reference temperature. The transport equation is a function of space, time and an angular variable. The source is represented by a unit step function.The **Laplace transform** is applied to the temporal or time variable which is defined in terms of radiation, temperature and material constants and the **Fourier transform** is applied to the space variable. A numerical routine based on Fortran code for a Romberg method is used to evaluate the resulting double integrals. The transport results are compared with a **Diffusion approximation** (Su and Olson 1996).The abstract is included along with the plots and a table from *The Annals of Nuclear Energy*. It is noticed that if the **Boltzmann Transport** results use a non-zero Intensity for t =0.0 microseconds with a scale factor for values of Intensity in the interval of [0,15] microseconds excellent agreement exist with the plots for Radiation energy density vs position and the Boltzmann transport theory scaled result. A plot is included using 6 microseconds and a non-zero Intensity at t=0.0 microseconds from the Boltzmann Transport results to be compared with a plot of Radiation energy density vs position based on data presented by Su and Olson.

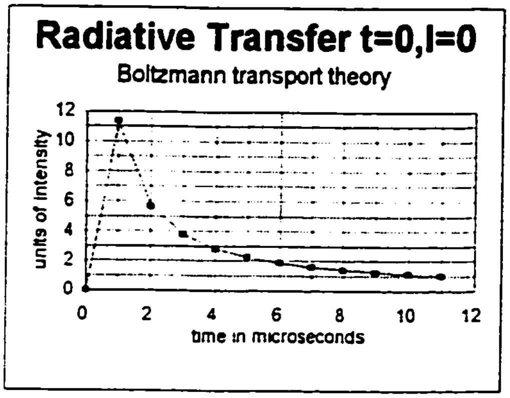


Figure 1 Standard result using zero intensity for t=0 microseconds.

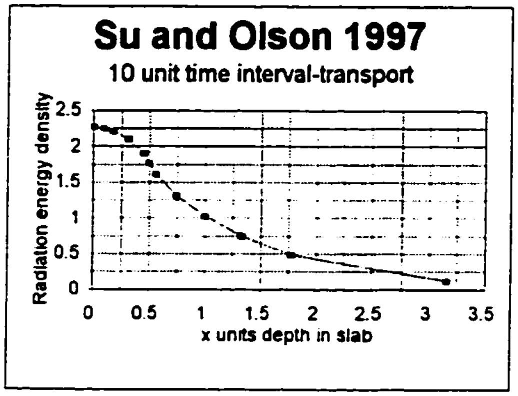


Figure 2 Su and Olson transport theory result based on data provided.

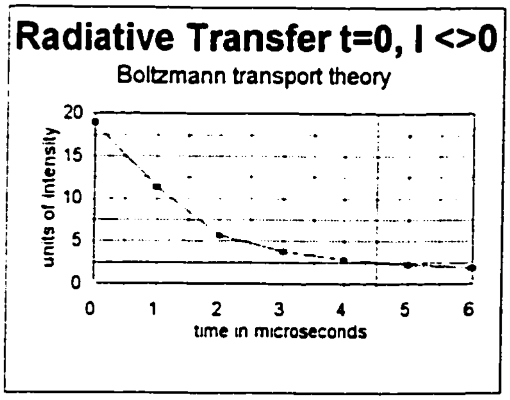


Figure 3 Standard plot replacing t=0 with a non-zero intensity.

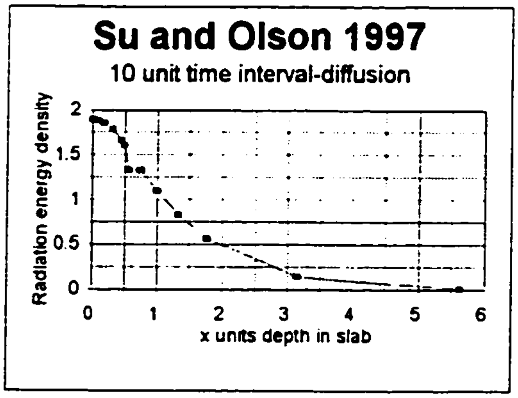


Figure 4 Su and Olson diffusion approximation result based on data provided.