

**A Preliminary Study of the Structural Dynamic Behavior
of the NASA Manned Spacecraft Center (MSC) Centrifuge**

by

Frederick W. Palmieri

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of the NASA Manned Spacecraft Center (MSC) Centrifuge*

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**A PRELIMINARY STUDY OF THE STRUCTURAL
DYNAMIC BEHAVIOR OF THE NASA MANNED
SPACECRAFT CENTER (MSC) CENTRIFUGE**

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF CIVIL ENGINEERING
OF MADISON UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Frederick William Palmieri
August 22, 2003

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor Of Philosophy.

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Approved by the Graduate Committee of the University.

Abstract

This dissertation involves a preliminary study into the structural dynamic behavior of the NASA Manned Spacecraft Center (MSC), located in the Flight Acceleration Facility, bldg 29, in Houston, Texas. The 50-ft. arm can swing the three-man gondola to create g-forces astronauts will experience during controlled flight and during reentry. The centrifuge was designed primarily for training Apollo astronauts. During operation of the centrifuge, the astronauts can control the motion of the gondola in two gimbal axes, while the gondola is rotating about its principal axis, to simulate flight activity. The result of these coupled motions lead to transient loading functions, which arise due to rigid body kinematics.

The study is describe in three Chapters. Chapter 1 deals with the response of a simplified model of the arm, gimbal and gondola structure for the purpose of obtaining dynamic response factors to be associated with the arm. Chapter 2 deals briefly with a simplified model of the same system for the purpose of obtaining dynamic response factors to be associated with the gimbal ring and to justify the simplifications implicit in the model used in Chapter 1. In Chapter 3, the rigid body kinematic equations are studied in order to develop relations between the forcing functions utilized in Chapters 1 and 2 and the motion parameters of the kinematic analysis. Using these relations, the dynamic response factors tabulated in Chapters 1 and 2 in terms of the generalized forcing functions may be interpreted in terms of the motion parameters.

The following assumptions have been made in order to obtain a solution within a reasonable time period for the preliminary study:

1. The gondola will be considered to be a rigid body;

2. The gimbal ring will also be considered to be a rigid body in the analysis of Chapter 1, and will be considered to be a simple spring-mass system in the analysis of Chapter 2;
3. The arm will be studied as another simple spring-mass system, acting as a uniform cantilever with a mass at its tip;
4. Small deflection, linear theory will be employed throughout the analysis;
5. Structural damping will be ignored for conservatism;
6. Lateral (tangential), vertical and torsional modes will be considered to be uncoupled and studied separately.

In defense of the apparent over-simplification of the complex system, which would be avoided if time permitted, it is to be observed that the simplifications are not as restrictive as they appear. The reason is that: 1) both the gondola and the gimbal ring are relatively rigid compared to the arm. Since the forcing functions have time durations that are not common in periodicity with the more rigid system but more common with the arm, simplification is permissible (as shown in Chapter 2 and Appendix 6); 2) The lumped parameter method is a conventional method of analysis; 3) Structural damping is small and the effect on transient motion is thus negligible; 4) An analysis (Appendix 3) has shown that coupling between modes does not occur; and 5) Neglecting the extensional mode is a conventional assumption and dynamic augmentation of centrifugal force is negligible.

The preliminary study of the structural dynamic response of the MSC centrifuge to transient loads resulting from gimbal-controlled motions has been completed. The results of the study are summarized in tables of dynamic response factors for lateral, torsional and vertical modes for three types of generalized impulsive loading functions: 1) a square pulse, 2) a saw-tooth ramp, and 3) a

half sine pulse. In Chapter 3 of the study, the rigid body kinematic equations have been analyzed so that these generalized loading functions used in the analysis may be interpreted in terms of the motion parameters.

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Nomenclature

Acronyms

MSC Manned Spacecraft Center

Roman Symbols

A,B,C,D,E	Constants
F	Forcing Functions
I	Moment of Inertia
i and j	Subscripts
K	Spring Constant
L	Laplace Transform
M or m	Mass or Subscript for Mass
o	Subscript for Time Constant
r	Subscript Denoting Arm
s	Laplacian Operator
t	Time
T	Torque or Moment
X or x	Torsional Axis or Subscript for Longitudinal Axis

Y or y	Lateral Axis or Subscript for Lateral axis
Z or z	Vertical Axis or Subscript for Vertical Axis

Greek Symbols

α	Flexibility Coefficient
β	Integration Variable
γ	Dynamic Response Factor
δ	Displacement
Δ	Characteristic Determinant
Θ	Roll Angle
Ψ	Yaw Angle and Yaw Axis
Φ	Pitch Angle
ζ	Eigenvalue
ω	Frequency
Θ_x	Angle about Torsional Axis
Θ_y	Angle about Roll Axis
Θ_z	Angle about Yaw Axis

Chapter 1 Introduction

1.1 Overview

This preliminary study is intended as an investigation into the structural dynamic behavior of the National Aeronautics and Space Administration (NASA) Manned Space Center (MSC) centrifuge used for training astronauts. In particular, the dynamic response to transient forcing functions that occur under imposed gondola rotations during astronaut training is predicted in the study.

The study is in three parts. Part A investigates the response of a simplified model of the arm, gimbal and gondola structure for the purpose of obtaining dynamic response factors to be associated with the arm. Part B deals briefly with a simplified model of the same system for the purpose of obtaining dynamic response factors to be associated with the gimbal ring and to justify the simplifications implicit in the model employed in Part A. In Part C, the rigid body kinematic equations are studied in order to obtain relations between the forcing functions used in Parts A and B and the motion parameters of the kinematic analysis. Using these relations, the dynamic response factors tabulated in Parts A and B in terms of the generalized forcing functions may be interpreted in terms of the motion parameters.

The following assumptions have been made in order to obtain a solution within the time available for the preliminary study:

1. The gondola will be considered to be a rigid body;
2. The gimbal ring will also be considered to be a rigid body in the analysis of Part A and it will be considered to be a simple spring-mass system in the analysis of Part B;
3. The arm will be considered to be a simple spring-mass system, acting as a uniform cantilever with a portion of its mass located at the tip of the arm;
4. Small deflection, linear theory will be employed throughout the analysis;
5. Structural damping will be ignored for conservatism;

6. Lateral (tangential), vertical and torsional modes will be considered to be uncoupled and will be studied separately.

In defense of the apparent over-simplification of the system, which would be avoided in a study with unlimited time, it is to be observed that the simplifications are not as restrictive as they appear.

The following factors must be considered in this regard:

1. Both the gondola and the gimbal ring are relatively rigid compared to the arm.
2. Since the forcing functions have time durations, which are not in common with the periodicity with the more rigid systems, but rather are in common with the period of the arm, simplification of the system is permissible (see Part B and Appendix 6 for a proof).
3. The lumped parameter technique is a conventional method of analysis and has been shown to be fairly accurate for the prediction of the response to transient forces.
4. An analysis (see Appendix 3) has shown that no coupling exists between the various modes of the simplified model during free vibration. A similar conclusion holds for the forced response if the forcing functions are considered to be independent of the secondary motions. Although it is possible that self-induced vibrations may occur as a result of coupling between kinematic forces and structural vibrations, an investigation of this phenomenon is beyond the scope of this preliminary investigation.
5. Neglecting the extensional mode is a conventional assumption. In the structure under investigation the dynamic augmentation of maximum centrifugal forces is negligible.

1.2 Summary

A preliminary study of the structural dynamic response of the NASDA-Houston MSC centrifuge to transient loads induced by rotations of the gondola during rotation of the centrifuge arm has been completed.

The results of the study are summarized in the “Tables of Dynamic Response Factors” for the arm for lateral, vertical and torsional modes for three types of typical impulsive, generalized forcing functions. They are:

1. Square step (better described as a rectangular pulse).
2. Ramp.
3. Half Sine.

In part C of the study, the rigid body kinematic equations have been analyzed so that the generalized loading functions employed in the analysis may be interpreted in terms of the motion parameters of the kinematic analysis.

Chapter 2 Arm Response

2.1 Overview

In this Chapter of the study the equations of motion derived in Appendix 1 are summarized and then utilized to determine the fundamental frequencies of free vibration. Then, the response of the gimbal arm, resulting from enforced transient motions of the gondola, is determined.

Figure 2.1.1 is a depiction of the centrifuge structure and Figure 2.1.2 shows a corresponding view of the idealized model of the structure.

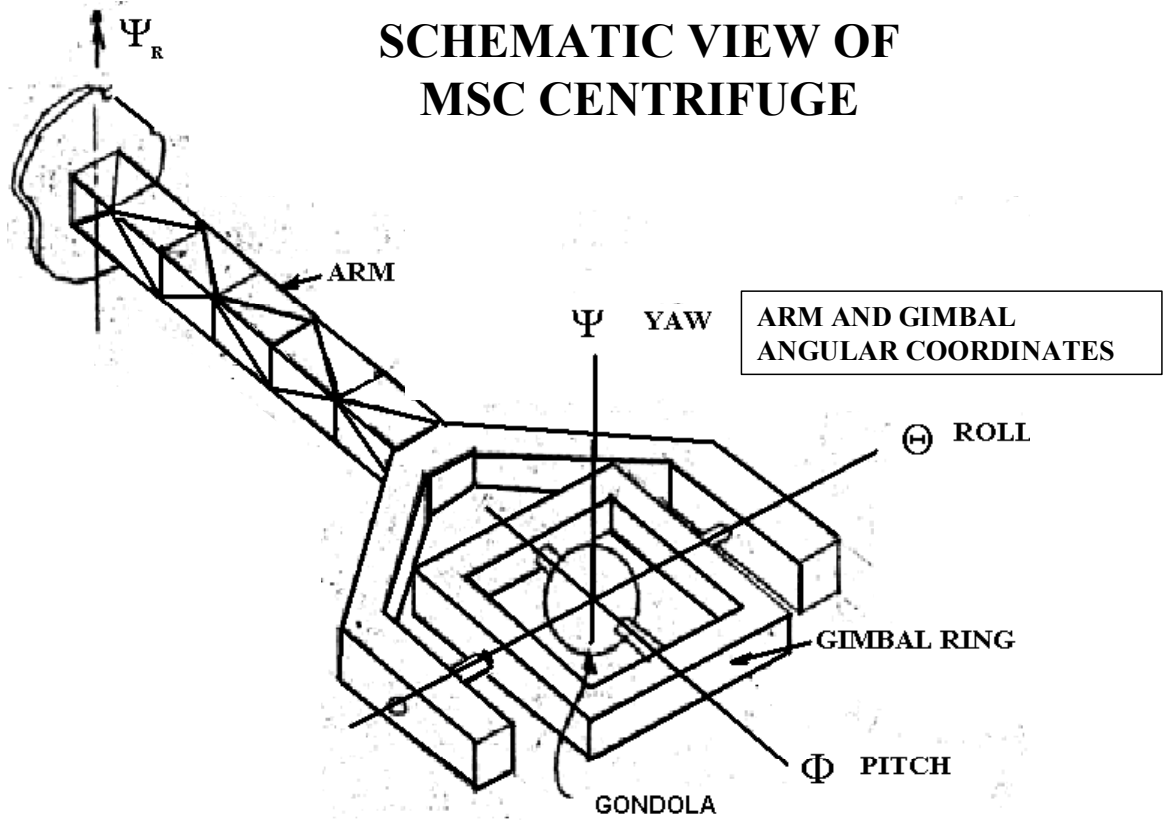


FIGURE 2.1.1

IDEALIZED MATHEMATICAL MODEL

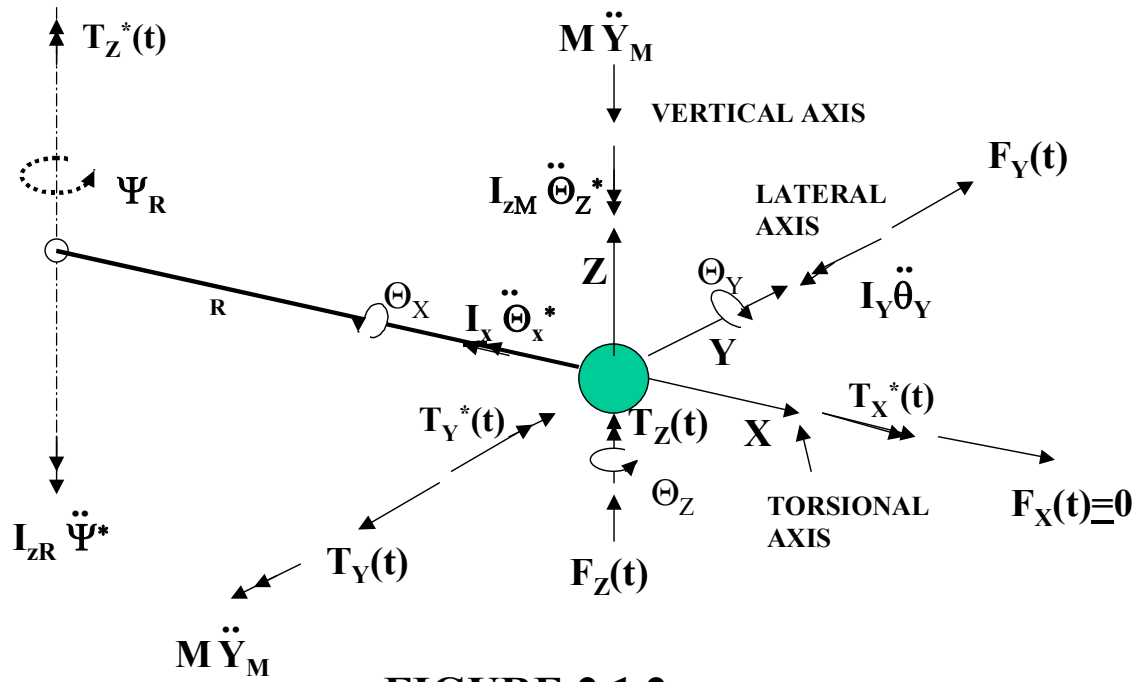


FIGURE 2.1.2

2.2 Summary – Equations of Motion

Y Axis:¹

$$\left\{ \begin{array}{l} MR\ddot{y}_M + I_{Z_M}\ddot{\theta}_Z + I_{Z_R}\ddot{\psi}^* = T_Z^*(t) + RF_y(t) + T_Z(t) \\ y_M - R\psi^* + \alpha_{11}M\ddot{y}_M + \alpha_{15}UI_{Z_M}\ddot{\theta}_Z = \alpha_{11}F_y(t) + \alpha_{15}T_Z(t) \\ \theta_Y + \psi^* + \alpha_{15}M\ddot{y}_M + \alpha_{55}I_{Z_M}\ddot{\theta}_Z = \alpha_{15}F_y(t) + \alpha_{55}T_Z(t) \end{array} \right\} \quad (9)'$$

Equations for the two other axes may be written directly:

Z Axis:

$$\left\{ \begin{array}{l} Z_M + \alpha_{22}M\ddot{Z}_M + \alpha_{24}I_{Y_M}\ddot{\theta}_Y = \alpha_{22}F_Z(t) + \alpha_{24}T_Y(t) \\ \theta_Y + \alpha_{42}M\ddot{Z}_M + \alpha_{44}I_{Y_M}\ddot{\theta}_Y = \alpha_{42}F_Z(t) + \alpha_{44}T_Y(t) \end{array} \right\} \quad (10)$$

X Axis:

$$\left\{ \theta_X + \alpha_{33}I_{X_M}\ddot{\theta}_X = \alpha_{33}T_X(t) \right\} \quad (11)$$

Where the α_{ij} represent flexibility coefficients. (See Appendix 2)

2.3 Y –Axis Analysis

2.3.1 Solution for Y axis normal modes

Assuming a solution to the free vibration problem in the form:

$$\left\{ \begin{array}{l} y_M = A_1 \sin \omega t \\ \theta_Z = A_2 \sin \omega t \\ \psi^* = A_3 \sin \omega t \end{array} \right\}$$

We obtain the following set of equations from (9):

$$\begin{aligned} -MR\omega^2 A_1 - I_{Z_M}\omega^2 A_2 - I_{Z_R}\omega^2 A_3 &= 0 \\ (1 - \alpha_{11}M\omega^2)A_1 - \alpha_{15}I_{Z_M}\omega^2 A_2 - RA_3 &= 0 \\ -\alpha_{15}M\omega^2 A_1 + (1 - \alpha_{55}I_{Z_M}\omega^2)A_2 - A_3 &= 0 \end{aligned}$$

¹ See Appendix 1 for derivation

Rearranging and using matrix notation:

$$\begin{bmatrix} \frac{1}{\omega^2} - a_{11} & a_{12} & \frac{1}{\omega^2} - a_{13} \\ a_{21} & \frac{1}{\omega^2} - a_{22} & \frac{1}{\omega^2} - a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = 0 \quad (12)$$

$$\text{Where, } \begin{cases} a_{11} = \alpha_{11}M; & a_{12} = -\alpha_{15}I_{Z_M}; & a_{13} = -R \\ a_{21} = \alpha_{15}M; & a_{22} = \alpha_{55}I_{Z_M}; & a_{23} = -1 \\ a_{31} = MR; & a_{32} = I_{Z_M}; & a_{33} = I_{Z_R} \end{cases}$$

The characteristic equation is given by the expansion of the determinant of the coefficient matrix set equal to zero. Thus,

$$\begin{aligned} & \left(\frac{1}{\omega^2} - a_{11} \right) \left(\frac{1}{\omega^2} - a_{22} \right) a_{33} + \frac{1}{\omega^2} (a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}) - \left(\frac{1}{\omega^2} - a_{22} \right) \left(\frac{1}{\omega^2} a_{13} \right) a_{31} \\ & - \left(\frac{1}{\omega^2} - a_{11} \right) \left(\frac{1}{\omega^2} a_{23} \right) a_{32} - a_{12}a_{21}a_{33} = 0 \end{aligned}$$

or,

$$\begin{aligned} & \left(\frac{1}{\omega^2} \right)^2 [a_{33} - a_{13}a_{31} - a_{23}a_{32}] + \\ & \left(\frac{1}{\omega^2} \right) [-a_{11}a_{33} - a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32}] - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} = 0 \end{aligned}$$

$$b_1 = a_{33} - a_{13}a_{31} - a_{23}a_{32}$$

$$\text{Denoting, } b_2 = -a_{11}a_{33} - a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32}$$

$$b_3 = -a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33}$$

$$\text{Then: } \left(\frac{1}{\omega^2} \right) + A \left(\frac{1}{\omega^2} \right) + B = 0 \quad \text{where, } \begin{aligned} A &= b_2/b_1 \\ B &= b_3/b_1 \end{aligned}$$

$$\text{Denoting } Z = \frac{1}{\omega^2}$$

$$\text{We find that } Z = -\frac{A}{2} \pm \sqrt{\left(\frac{A}{2} \right)^2 - B}$$

Letting $Z_1 > Z_2$ where Z_1 and Z_2 are the characteristic roots, we have the natural

$$\text{frequencies: } \left\{ \begin{array}{l} \omega_1 = Z_1^{-1/2} \\ \omega_2 = Z_2^{-1/2} \end{array} \right\} \quad (13)$$

The displacement ratios for each mode can be determined as follows:

$$\text{Let } A_1 = 1, \quad \text{for } \omega = \omega_i$$

Then, from eqs. (12),

$$\begin{aligned} \frac{1}{\omega_i^2} - a_{11} + a_{12}A_2 + \frac{1}{\omega_i^2}a_{13}A_3 &= 0 \\ a_{21} + \left(\frac{1}{\omega_i^2} - a_{22} \right)A_2 + \frac{1}{\omega_i^2}a_{23}A_3 &= 0 \\ a_{31} + a_{32}A_2 + a_{33}A_3 &= 0 \end{aligned}$$

$$\text{From the last Eq. ,} \quad A_{2,i} = -\frac{a_{33}}{a_{32}}A_{3,i} - \frac{a_{31}}{a_{32}} \quad (14)$$

Substitute into the second Eq. :

$$a_{21} + \left(\frac{1}{\omega_i^2} - a_{22} \right) \left(-\frac{a_{33}}{a_{32}}A_3 - \frac{a_{31}}{a_{32}} \right) + \frac{1}{\omega_i^2}a_{23}A_3 = 0$$

$$\text{or, } a_{21} - \left(\frac{1}{\omega_i^2} - a_{22} \right) \frac{a_{31}}{a_{32}} = A_3 \left[-\frac{a_{23}}{\omega_i^2} + \left(\frac{1}{\omega_i^2} - a_{22} \right) \frac{a_{33}}{a_{32}} \right] = 0$$

$$\text{or, } A_{3,i} = \frac{a_{21}a_{32} - (Z_i - a_{22})a_{31}}{(Z_i - a_{22})a_{33} - a_{23}a_{32}Z_i}$$

$$A_{3,i} = \frac{a_{21}a_{32} + a_{22}a_{31} - a_{31}Z_i}{(a_{33} - a_{23}a_{32})Z_i - a_{22}a_{33}} \quad (15)$$

$$\text{Of course, } A_{1,i} = 1 \quad (16)$$

Equations (14), (15), and (16) then define the displacement ratios.

2.3.2 Solution For Y Axis Transient Response

Knowing the normal modes and frequencies of the system, the solution for the transient motions can be obtained in terms of these modes. Using a convenient numerical procedure (known as the phase-plane-delta method) where the forcing functions are complex time functions.

When the forcing functions are simple², a closed form solution using operation methods is possible.

Eqs. (9) are operated on by the Laplacian operator $s = \frac{d}{dt}$

Let:

$$\left\{ \begin{array}{l} L\{y_M(t)\} = u_1(s) \\ L\{\theta_Z(t)\} = u_2(s) \\ L\{\psi^*(t)\} = u_3(s) \end{array} \right\} \quad \left\{ \begin{array}{l} L\{T_Z^*(t)\} = f_1(s) \\ L\{F_y(t)\} = f_2(s) \\ L\{T_Z(t)\} = f_3(s) \end{array} \right\}$$

Assume:

$$\left\{ \begin{array}{l} \dot{y}_0 = \theta_{Z_0} = \psi_0 = 0 \\ \dot{y}_0 = R\dot{\psi}_0^* \\ \dot{\theta}_{Z_0} = \dot{\psi}_0^* \end{array} \right\}$$

Operating on Eqs. (9) : $L(\ddot{x}) = s^2 L(x) - sx_0 - \dot{x}_0$

Loading

$$\left\{ \begin{array}{l} MR(s^2 u_1 - R\dot{\psi}_0^*) + I_{Z_M}(s^2 u_2 - \dot{\psi}_0^*) + I_{Z_R}(s^3 u_3 - \dot{\psi}_0^*) = L[\dots] \\ u_1 - Ru_3 + \alpha_{11}M(s^2 u_1 - R\dot{\psi}_0^*) + \alpha_{15}I_{Z_M}(s^2 u_2 - \dot{\psi}_0^*) = L[\dots] \\ u_2 - u_3 + \alpha_{15}M(s^2 u_1 - R\dot{\psi}_0^*) + \alpha_{55}I_{Z_M}(s^2 u_2 - \dot{\psi}_0^*) = L[\dots] \end{array} \right\} \quad (17)$$

In matrix notation,

$$\begin{bmatrix} 1 + \alpha_{11}Ms^2 & -\alpha_{15}I_{Z_M}s^2 & -R \\ -\alpha_{15}Ms^2 & 1 + \alpha_{55}I_{Z_M}s^2 & -1 \\ MRs^2 & I_{Z_M}s^2 & I_{Z_M}s^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f_1(s) + \begin{bmatrix} R \\ \alpha_{11} \\ \alpha_{15} \end{bmatrix} f_2(s) + \begin{bmatrix} 1 \\ \alpha_{15} \\ \alpha_{55} \end{bmatrix} f_3(s) \quad (18)$$

$$+ \begin{bmatrix} MR^2 + I_{Z_M} + I_{Z_R} \\ \alpha_{11}MR + \alpha_{15}I_{Z_M} \\ \alpha_{15}MR + \alpha_{55}I_{Z_M} \end{bmatrix} \dot{\psi}_0^* = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

² Simple => Restricted to functions of class "A".

Using the notation of Eqs. (12), we have

$$\begin{bmatrix} 1 + \alpha_{11}s^2 & -a_{12}s^2 & a_{13} \\ -a_{21}s^2 & 1 + a_{22}s^2 & a_{23} \\ a_{31}s^2 & a_{32}s^2 & a_{33}s^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (19)$$

Using Cramer's Rule:

$$u_1 = \frac{\begin{vmatrix} P_1 & -a_{12}s^2 & a_{13} \\ P_2 & 1 + a_{22}s^2 & a_{23} \\ P_3 & a_{32}s^2 & a_{33}s^2 \end{vmatrix}}{\Delta}$$

$$u_1 = \frac{P_1[(1 + a_{22}s^2)(a_{33}s^2) - a_{23}(a_{32}s^2)]}{\Delta} + \frac{P_2[a_{12}a_{33}s^4 + a_{13}a_{32}s^2]}{\Delta}$$

$$+ \frac{P_3[-a_{12}a_{23}s^2 - (1 + a_{22}s^2)a_{13}]}{\Delta} \quad (20)$$

$$u_2 = \frac{\begin{vmatrix} 1 + a_{11}s^2 & P_1 & a_{13} \\ -a_{21}s^2 & P_2 & a_{23} \\ a_{31}s^2 & P_3 & a_{33}s^2 \end{vmatrix}}{\Delta}$$

$$= P_1 \frac{[a_{21}a_{33}s^4 + a_{23}a_{31}s^2]}{\Delta} + P_2 \frac{[(1 + a_{11}s^2)a_{33}s^2 - a_{13}a_{31}s^2]}{\Delta}$$

$$+ P_3 \frac{[-(1 + a_{11}s^2)a_{23} - a_{13}a_{21}s^2]}{\Delta} \quad (21)$$

$$u_3 = \frac{\begin{vmatrix} 1 + a_{11}s^2 & -a_{12}s^2 & P_1 \\ -a_{21}s^2 & 1 + a_{22}s^2 & P_2 \\ a_{31}s^2 & a_{32}s^2 & P_3 \end{vmatrix}}{\Delta}$$

$$= P_1 \frac{[-a_{21}a_{32}s^4 - (1 + a_{22}s^2)a_{31}s^2]}{\Delta} + P_2 \frac{[-(1 + a_{11}s^2)a_{32}s^2 - a_{12}a_{31}s^4]}{\Delta}$$

$$+ P_3 \frac{[(1 + a_{11}s^2)(1 + a_{22}s^2) - a_{12}a_{21}s^4]}{\Delta} \quad (22)$$

Where $\Delta =$ characteristic determinant $= p(s)$