SUPER EDGE-ANTIMAGIC GRAPHS

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A Wealth of Problems and Some Solutions

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Preface

In writing this monograph we have set out to accomplish several goals. We are aiming to present a small part of the fascinating subject of graph labeling, called antimagic. In this we have been inspired and motivated by Wal Wallis who has published "Magic Graphs", the first book on magic graph labeling. While the terminology may suggest that "antimagic" is the opposite of "magic", this is not so, and in fact, as the reader will soon find out, our kind of antimagic labeling is a generalisation of magic labeling; and magic graphs are a subset of antimagic graphs.

We have tried to make our book as comprehensive and as self contained as possible. However, we do recommend the reader to consult also the excellent dynamic survey by Joe Gallian whose life work has been a great inspiration to many new young researchers, as well as a most useful reference to the already converted "graph labelists".

We have also tried to make our exposition clear and relatively simple so that it can be easily picked up by anybody with a good foundation in high school mathematics; and nothing more.

Graph theory, and graph labeling in particular, are fast growing research areas. New results are being discovered and published at a rapidly increasing rate. There is an enormous number of open problems and conjectures in graph labeling; some of these have been included in this book. We hope that this book will not only be useful to existing researchers in this area but that it will also contribute to inspire new researchers to "give graph labeling a go".

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June 2008

Contents

List of Figures

Chapter 1

Introduction

A *graph* G is an ordered pair of sets $G = (V, E)$, where V is a set of entities, typically called *points*, or *nodes*, or *vertices*. In this monograph, we will be interested only in finite graphs. The set of points V can be used to represent all kinds of objects, real or imaginary, in the "real" world, such as computers, people, cities, and many more. The second set, E, represents a binary relation between the points, for example, connections between computers (there is a connection between two computers or not), kinship between people (two people are known to be related or not), roads between cities (there is a road between two cities or not), and so on. The elements of the set E are usually called *edges*. While it is entirely possible to deal with a graph as a mathematical structure $G = (V, E)$, graphs also have a natural pictorial representation: V is drawn as an arbitrary scattering of points, typically on a plane, each point representing an element of V, and E is drawn as a set of lines between the points; a line can be drawn straight or curved but never crossing itself. This pictorial representation of graphs is very helpful in modelling of various real world situations and problems as graphs, as well as in aiding researchers in their quest to find new general theoretical results concerning graphs, results, which can subsequently become useful in applications.

While the basic notion of a graph, as described above, seems to be simple, there are many interesting unsolved problems, mostly related to the distances between points, measured by the number of edges that need to be traversed in a shortest path between two points, and/or related to the number of edges and their distribution in the graph. Such unsolved problems are typically very difficult.

To make general graph theoretical problems even more difficult, we can add direction to the edges to obtain a *directed graph*, a *digraph*, for short. Such a generalization is dictated by applications; for example, digraphs can be used to model one-way traffic in a city. In this monograph we do not deal with digraphs.

Most modifications of the basic notion of a graph tend to be motivated by applications. For example, when dealing with the famous Travelling Salesman Problem, we add weights to the edges, indicating the distance between the cities that are represented as the endpoints of an edge. As another example, in the Chinese Postman Problem, we also add weights to the vertices, designating the population of a destination represented by a vertex. There are many more modifications to the basic idea of a graph that are used to fit a particular application.

In this book, we deal with specialized graphs that are obtained by labeling the edges, or vertices, or both, in a specific way that satisfies certain conditions.

The concept of labeling of graphs has recently gained a lot of popularity in the area of graph theory. This popularity is due not only to the mathematical challenges of graph labelings but also to the wide range of applications that graph labelings offer to other branches of science, for instance, x-ray, crystallography, coding theory, cryptography (secret sharing schemes), astronomy, circuit design and communication networks design [36], [37].

Informally, by a graph labeling we will mean an assignment of integers to elements of a graph, such as vertices, or edges, or both, subject to some specified conditions. These conditions are usually expressed on the basis of the values (called *weights*) of some evaluating function. In our case, the evaluating function will be simply to produce partial sums of the labeled elements of the graph. The partial sums will be either a (multi)set of *vertex weights*, obtained for each vertex by adding all the labels of a vertex and its adjacent edges, or a (multi)set of *edge weights*, obtained for each edge by adding the labels of an edge and its endpoints.

One of the situations that we are particularly interested in is when all the edge weights or all the vertex weights are the same. In such a case we call the labeled graph *edge magic* or *vertex magic*, respectively. Magic graphs, edge or vertex, or even simultaneously edge and vertex magic (called *totally magic*) are described in the pioneering book by Wal Wallis.

Another situation that is of interest is when all the edge weights or all the vertex weights are different. In such a case we call the labeled graph *edge antimagic* or *vertex antimagic*, respectively. The study of these graphs was motivated by Nora Hartsfield and Gerhard Ringel, who considered labeling uniquely the edges of a graph containing q edges using the integers 1,2,...,q, and evaluating partial sums of labels at the vertices of the graph. If all the vertex weights are different then they call the graph *antimagic*. Immediately, it is easy to see that a graph consisting of two vertices and an edge between them (denoted K_2) cannot be labeled antimagically. However, this is likely to be the only such graph and Hartsfield and Ringel propose the following conjecture.

Conjecture 1.0.1. [73]. *Every connected graph different from K_2 is antimagic.*

This conjecture is still open. A *tree* is a connected graph that does not contain any cycles. The graph K_2 can be regarded as a tree on two vertices. Interestingly, even if we restrict ourselves to trees, it is not known whether the above conjecture is true. Therefore, Hartsfield and Ringel also posed the following weaker conjecture.

Conjecture 1.0.2. [73] *Every tree different from K_2 is antimagic.*

While trying to prove the conjecture, it was soon realised that not only every graph considered had an antimagic labeling, but moreover, there are usually many different antimagic labelings. So we have a very interesting situation: on one hand, it seems difficult to prove the Hartsfield and Ringel conjecture or to find a counterexample, and on the other hand, most graphs seem to admit a large number of labelings that are antimagic.

This situation has led to the idea of placing some more restrictions on antimagic labelings. In particular, Bodendiek and Walther [38] introduced the idea of an antimagic labeling with the condition that not only are all the vertex weights different, moreover, they require that the weights form an arithmetic progression.

These antimagic labelings can be referred to as *vertex antimagic labelings*. Subsequent variations of this concept include *super edge antimagic labelings* which are the main topic of this monograph.

This book begins with a short overview of magic and edge antimagic labelings, in Chapters 2 to 4. The main topic of the book, super edge antimagic labeling, is presented in Chapters 5 to 12. At the end of the book, there is a list of open problems and conjectures, and also an extensive Bibliography.

We wish to thank the following friends and collaborators for many enjoyable and valuable discussions about various magic and antimagic graph labelings: Camino Balbuena, Christian Barrientos, Ewan Barker, Edy Tri Baskoro, Francois Bertault, Gary Bloom, Ljiljana Brankovic, Yus Cholily, Dafik, Dalibor Fronček, S.M. Hegde, Ivan Hollånder, Jaroslav Ivančo, Stanislav Jendrol̆, Petr Kovář, Tereza Kovářová, Yuqing Lin, Anna Lladó, Jim MacDougall, Francesc A. Muntaner-Batle, Muthali Murugan, Alex Rosa, Joe Ryan, Andrea Semaničová, Rinovia Simanjuntak, Slamin, Kiki A. Sugeng, Michal Tkáč, Marián Trenkler, Maged Z. Youssef, and Wal Wallis.

Our special thanks go to Andrea Semaničová for redrawing and improving all the figures.

However, the authors alone should be blamed for any remaining errors and imperfections.

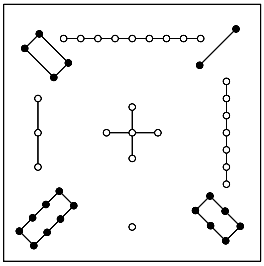
Chapter 2

Preliminaries

We denote by $V(G)$ the set of vertices and by $E(G)$ the set of edges of a graph $G(V, E)$. Let $|V(G)| = p$ and $|E(G)| = q$ be respectively the number of vertices and the number of edges of G. When convenient, we sometimes refer to G as a (p, q)-*graph*. General reference for graph-theoretic notions is [139].

By a *labeling* we mean a one-to-one mapping that carries a set of graph elements into a set of numbers (usually integers), called *labels*. We deal with labelings with domain either the set of all vertices, or the set of all edges, or the set of all vertices and edges, respectively. We call these labelings a *vertex labeling*, or an *edge labeling*, or a *total labeling*, depending on the graph elements that are being labeled.

The concept of labeling of graphs has gained a lot of popularity in the area of graph theory. This popularity is not only due to mathematical challenges of graph labelings but also to the wide range of applications that graph labelings offer to other branches of science, for instance, x-ray, crystallography, coding theory, cryptography (secret sharing schemes), astronomy, circuit design and communication networks design [36], [37].



4	9	2
3	5	7
8	1	6

Figure 2.1: Lo Shu magic square.

2.1 Magic squares

In recreational mathematics, a *magic square* of order n is an arrangement of n^2 numbers, usually distinct integers, in a square pattern, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant, known as the *magic constant*. A *normal magic square* contains the integers from 1 to n^2 (see [6] and [100]). The magic constant of a normal magic square of order n has the value $\lambda = \frac{n(n^2+1)}{2}$.

The first normal magic square known to be recorded was a square of order 3 called *Lo Shu* around 2200 BC. According to the legend about the Chinese Emperor Yu, from the book Yih King, the diagram was found on the shell of a divine turtle (see [91]). Figure 2.1 depicts the Lo Shu magic square and the corresponding normal magic square.

The first record of a magic square in Europe is found on a famous engraving *Melencolia I* from 1514 by the German artist and scientist Dürer. The Dürer's normal magic square of order 4 is very similar to Yang Hui's square, which was created in China about 250 years before Dürer's time. The magic constant $\lambda(4) = 34$ can be found in the rows, columns, diagonals, each of the quadrants, the center four square and the corner square. The two numbers in the middle of the bottom row give the date of the engraving. Dürer's engraving *Melencolia I* can be seen in [54] and Figure 2.2 shows the Dürer's magic square.

From historical facts, we can see that the (normal) magic square has existed throughout history and in many different parts of the world. Mathematics is

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.2: Dürer's magic square on engraving *Melencolia I.*

all around us and our mind will see it whenever we are ready.

Normal magic squares exist for all orders $n \geq 3$. There are many ways to construct magic squares, but the standard way is to follow certain configurations which generate regular patterns. The book of Andrews [6] is probably the definitive work on magic squares. It shows how to construct normal magic squares and all of the many variations that exist. The book is highly technical and of more interest to the serious mathematician than the average magician.

It is an unsolved problem to determine the number of non isomorphic normal magic squares of an arbitrary order. For $n = 3$, there is only one normal magic square. The 880 normal magic squares of order $n = 4$ were enumerated by Frénicle de Bessy (1693), and are illustrated in [34]. The number of normal magic squares of order $n = 5$ is 275 305 224 and it was computed by Schroeppel in 1973 (see [34]). The number of normal magic squares of order $n = 6$ is not known, but Pinn and Wieczerkowski [107] estimated it to be $(1.7745 \pm 0.0016) \times 10^{19}$, using Monte Carlo simulation and methods from statistical mechanics. Results of historical and computer enumeration of the number of non isomorphic normal magic squares can be found in [135].

2.2 Antimagic squares

An antimagic square is one of many variations of magic square. An *antimagic square* of order n is an arrangement of the numbers 1 to n^2 in a square, such