The Pendulum Paradigm

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Variations on a Theme and the Measure of Heaven and Earth

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This book is humbly dedicated to Robert Hooke (1635-1703),
a man who marveled at the rhythmical vibrancy of the universe,
and a man who never ceased in putting nature to the question.

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INTRODUCTION

The world is in perpetual turmoil and motion – it is a vibrant extravaganza of interactions, oscillations and pulses. Indeed, we have all witnessed the to and fro of nature's ever changing mood; the rhythmical swaying of leaf-dappled trees; the voluminous ripple of wheat stems in a gentle breeze; the furious beating of a tattered flag in a winter's gale; the stately rise and fall of ocean tides. Likewise the numinous 'tock' of a grandfather clock, repeating in a darkened room on a sleepy Sunday evening, reminds us of the steady ebb and flow of time. Movement surrounds us, and our lives dance to its ever-changing tune. Deep down, however, there is an underlying order, at least of sorts, and we can breakdown the complex motions of the everyday world into a summation of simple harmonic waves. Underpinning all this dynamical mêlée is the pendulum – perhaps the simplest experimental device ever invented. The pendulum for all its apparent simplicity, however, has been at the historical center of humanities exploration of the Earth and the sky above, and this book is about some of that remarkable history. Indeed, this book is concerned with the transformation of the pendulum from its early origins as an ungainly piece of string and an attached weight to that of a precision scientific instrument. Incredibly, for so it seems at first, by simply counting the number of oscillations, back and forth, the pendulum has acted as the key tool in the discovery of Earth's motion and its internal structure, and it has also enabled the experimental measure of the fundamental physical constants defining our universe.

More than just an intricate scientific device and timepiece regulator, however, the metaphorical pendulum is an integral part of our everyday language and lives; all politicians fear the swing of the ideological pendulum, and as Ralph Waldo Emerson so dryly put it, "Old age brings along with its ugliness the comfort that you will soon be out of it – which ought to be a substantial relief to such discontented pendulums as we are". And, indeed, we all hope that the pendulum will swing in our favor. The pendulum is a wonderful metaphor for the confined cycle of change; "what goes around comes around".

The story of the pendulum is primarily the story of observing small changes to quantify the very large – it literally displays the rift between the ideal and reality. It condenses the vastness of the Earth into laboratory-sized experiments which can be run on any weekday afternoon – as all undergraduate students of science well know. The pendulum additionally takes us from the local to the global – elevating our reach beyond the confines of the laboratory to the very edge of the universe. Indeed, as we shall see in Chapter 4 it was a pendulum experiment performed by Isaac Newton in his Cambridge University chambers that convinced him that there was no such element as the gravitational ether, or universal medium through which gravity is transmitted. Newton not only pinned down a formula to describe how gravity must work, therefore, he also showed (albeit non-rigorously by modern standards) that gravity works outside of any transmitting fluid.

And, while the exact workings of gravity are still a mystery to present day researchers, so too are the workings of time. Wryly, the composer Hector Berlioz was once heard to quip that, "time is a great teacher, but it unfortunately kills all its pupils". Sadly this is all too true, but for all this, at the core of time measurement and the pace of musical measure are the pendulum and the metronome (the latter, of course, being an inverted rigid pendulum). While less relevant to mensuration in the modern era, where atomic clocks rule supreme, the pendulum clock historically allowed astronomers to chart the heavens and measure the spin and shape of the Earth. Likewise, the innate human desire for consistency has resulted in the globe being divided into standard time zones; each swath 15 degrees wide in longitude, with every clock (pendulum regulated or otherwise) being forced to agree on the time decreed. And, all this standardization and international cooperation is centered on a concept that we still cannot explain. Time is a universal mystery; we don't know what it is, and we cannot define its origins. As Albert Einstein so rightly told us, almost a century ago now, "time is an illusion" – to which we humbly add, "but what a marvelous illusion". Indeed, through the power of commerce and the desire for order, humanity has managed to ensnare itself within the great illusion of time, and there appears to be little hope of our ever escaping from this overwhelming entrapment. Perhaps, as ever, playwright William Shakespeare found the right words to describe how it is,

I wasted time, and now doth time waste me;
For now hath time made me his numbering clock:
My thoughts are minutes; and with sighs they jar
Their watches on unto mine eyes, the outward watch,
Whereto my finger, like a dial's point,
Is pointing still, in cleansing them from tears.
Now sir, the sound that tells what hour it is
Are clamorous groans, which strike upon my heart,
Which is the bell: so sighs and tears and groans
Show minutes, times, and hours.

So speaks King Richard II, in the play of the same name. Earth has been encaged, and we are tied to time, that grand illusion which is rhythmically sliced by the scythe-like swing of the pendulum bob. And yet, time comes and goes, and according to our mood and surroundings it passes in a quick-silver shimmer or as slow as molasses – sometimes it even stops for a breathless moment. Time may be an illusion according to physics, but the pendulum is the physical manifestation of its task master.

The mighty pendulum, for so it seems worthy of this great appellation, has accompanied humanity on its quest to discover and annotate the greater world and universe. It is a device that is easy to contemplate, but is subtle in its power and long in its reach; it is an instrument that spans the chasms between human nature, time and space – mighty indeed is the pendulum, and remarkable, as we shall see in subsequent pages, is its story of revelation.

And finally, a few comments are probably in order with respect to what this book is intended to be about and what it is not. To begin with the latter, this book is not intended to be a comprehensive tome on the history of science; nor is it intended to be a mathematical or physical treatise. My inspiration, as much as I can remember it now, has been to write the sort of book that set my mind and imagination racing when I was an undergraduate student studying mathematics and physics in the mid-1970s at the University of Sussex, in Brighton England. Accordingly it is intended that this book serves as a review of the history of science with snippets of the physical principles and some of the mathematical details included (this is in contrast to the apparent norm where they are totally excluded). The material and topics covered are intentionally eclectic, broad and wide-ranging, and they will hopefully inspire the reader's imagination as much as they have, over the years, inspired the author's. In addition to being a review of the history and science relating to pendulum physics, my aim has also been to bring the topic to the present day – the pendulum may be one of humanities oldest of inventions, but it is also one of its most enduring. Far from being a device of the ancients only, the mighty pendulum is still one of the great tools of modern science with a future that is as bright as its history is long. The layout of the text is such that it need not be read in a linear fashion, but many of the chapters are interrelated – so dip-in or dig-deep as the mood takes you, but most of all it is my hope that you will enjoy and be inspired to look further into the stories that you find.

CHAPTER 1
THE PENDULUM SWINGS

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

J. von Neumann

Mention the word pendulum in the modern era and the mind immediately turns to Galileo, the great renaissance philosopher who revolutionized the way in which we think about the physical word. Galileo lead a charmed life for many years, playing with great panache and skill the complex game of individual advancement within 17th Century Italian society. His inability to suffer fools lightly, however, or as some would say his very arrogance, ultimately resulted in his fall from grace, and following several misadventures with Church authorities he was eventually forced, in 1633, to publicly recant his heresy of teaching, as true, the Copernican theory of the solar system. The Inquisition argued that Galileo should be jailed for the remainder of his natural life, but his sentence was commuted to that of house arrest. It was while confined to a comfortable home penitentiary, in Arcetri, just outside Florence, however, that Galileo produced his most enduring, if not most famous, work *Dialogues Concerning Two New Sciences* (published 1638). This work was based upon many decades of diligent labor, research, experimentation and intellectual reasoning, and it revolutionized the way in which subsequent scientific investigation would be conducted. The *Dialogues* is concerned with an imagined conversation between three interlocutors: Salviate, Sagredo and Simplicio. Lasting a total of four days, these three academics explore the possible explanations for numerous physical phenomena; it is the voice of Salviate, however, that always carries the argument, for indeed, Salviate is the strident voice of anti-Aristotelian reasoning. Galileo was unabashedly taking-on the intellectual establishment in his *Dialogues*, and he was fighting against the entrenchment of ancient, un-tested ideas. His voice, the voice of Salviate, was that of the new order, and the new experimentalist who carefully measured and reasoned without resort to unjustified assumption and dogma. The pendulum, in recognition of its great physical importance, is an object of discussion on days one, three and four within the *Dialogues*, and here is how it is first introduced:

Sagredo: *Another question deals with vibrations of pendulums which may be regarded from several viewpoints; the first is whether all vibrations, large, medium, and small, are* *performed in exactly and precisely equal times: another is to find the ratio of the times of vibrations of pendulums supported by threads of unequal length.*

Salviati: *These are interesting questions: but I fear that here, as in the case of all other facts, if we take up for discussion any one of them, it will carry in its wake so many other facts and curious consequences that time will not remain to-day for the discussion of all.*

Galileo cuts to the very core of our subject within his introductory, first-day dialog. Sagredo raises a number of important queries concerning the essential motion of the pendulum, and as Galileo, under the voice of Salviati, notes they are interesting questions that will take more than one day to address. Here, then, is our starting point. While this chapter won't take more than a day to read, we shall nonetheless attempt to unravel within its compass the long and fascinating story behind the mathematics of the pendulum, and we shall also see how pendulum experiments can be used to address questions relating to fundamental physics, metrology, geophysics, astronomy and even philosophy.

The physicist and moral philosopher Stephen Toulmin was once heard to wisely argue that, definitions are like belts, the shorter they are the more flexible they need to be. And, indeed, the standard definition of the pendulum, found in any dictionary, will require a little stretching-out at times. The word pendulum is etymologically derived from the Latin *pendulus*, meaning 'hanging', which is derived from *pendrere*, meaning 'to hang'. So, a pendulum is composed of a body (called 'the bob') of mass m, suspended from a fixed support by a wire or thread of length L in such a manner that it can swing freely, backwards and forwards, under the influence of gravity.

The dictionary style definition of a pendulum seems reasonable enough and agrees well with our everyday expectations. There is, however, some necessary unpacking, as Toulmin warned us there might be, of the definition supplied. Firstly, one might ask for all its triviality, why the suspended object is called a 'bob'; well, as with many such things, the name is a matter of history. There is an old saying which sagely notes that even a stopped clock (a mechanical one with hands that is) is right twice per day. Indeed, if such statements apply to the hands of a clock, then one might further argue, with some considerable degree of confidence, that a stationary pendulum is always right. Right, that is, if it is the local vertical that one wishes to find. The simple device of a plumb-line is still employed by construction workers to this very day, and consists of a heavy weight attached to a piece of string. The name plumb-line is derived from the Latin *plumbum*, meaning 'lead', which described the weight part of the line. This word for lead was later corrupted to plum bob and then cut down to the diminutive bob, and it is this final corruption that is now given to any weight attached to the end of a line used to measure the local vertical. Since a stopped pendulum is a plumb-line, the name bob fits naturally to the hanging weight, and as we shall see in later chapters there are many different uses for a stopped pendulum.

Perhaps more importantly for the analysis to follow, the dictionary definition for a pendulum should be expanded to say that the wire or thread supporting the bob is inextensible. A bob attached to an elastic suspension wire is a mathematical beast, albeit a highly interesting one, that will not be consider here (but see Chapter 3 later). Likewise, the definition should also be expanded to explain what is meant by the term 'mass', and for that matter what exactly does 'under the influence of gravity' mean. To see why such further expansion of our definition is required, we will develop in the next section a dimensional analysis formulation for the relationship that binds together the observable and physical properties of a pendulum.

Dimensional Analysis

The idea of dimensional analysis[1] surfaced in 1687 and it was first applied by the great mathematician and alchemist Isaac Newton (1643 - 1727). The use of dimensional analysis is a standard dodge still used to this very day by physicists and engineers alike. The point of dimensional analysis is to uncover a functional relationship (i.e., an equation) between the physical variables that enter into a specific problem by making sure that all of the units agree. The term 'physical variable' should first be explained before worrying about what a 'unit' is – although the two are closely linked. A physical variable describes a quantity, such as the length of the pendulum wire L, whose numerical value depends upon the units being used. For example, the distance $D = 1$ kilometer can be written as $D = 1000$ meters, or $D = 0.6213712$ miles, or $D = 39370.079$ inches, or $D = 1.057 \times 10^{-13}$ light years. In each case the distance D is exactly the same, but its numerical value changes according to the units being employed. In the discussion that follows *SI* or *Système International* units are going to be used. Commissioned at the request of King Louis XVI of France the *SI* units were first developed in the early 1790s. Indeed, on August 1st, 1793 the meter was introduced as the standard measure of length, and on April 7th, 1795 the gram and kilogram were adopted as standard measures for mass. The standard unit of time didn't change under the *SI* scheme, and it is taken to be the second – this being said, the second was only officially defined in the 1820s when it was agreed that it should correspond to 1/86,400th of a mean solar day. The meter is technically defined as the distance between two marks scribed on a platinum-iridium bar corresponding to one ten millionth (1 / 10,000,000) of the distance between the equator and the North Pole along the meridian passing through Paris. As is often the case with good intentions, however, there was, in spite of the heroic efforts of the surveyors and mathematicians, a miscalculation in the spacing of the two standard marks for the meter – they are in fact 1/5th of a millimeter too close together with respect to the original definition[2]. The miscalculation, however, has been allowed to stand. Standard units, after all, are ultimately just convenient measures that people (or more specifically Governments) either agree to use or they don't.

In spite of common language usage, whereby the words weight and mass are taken to mean the same thing, they are, in fact, very different physical quantities. Mass is defined in terms of the amount of matter that constitutes a given body. Specifically, the gravitational mass is a measure of the strength with which an object interacts with a gravitational field. Indeed, once the gravitational field is 'fixed' a smaller mass will experiences a smaller gravitational force than that experienced by a larger mass. The gravitational force associated with a specific mass defines its weight, and in physical terms the weight of an object is the product of its mass times the gravitational field strength g (gravitational field strength is expressed in terms of an acceleration and correspondingly has the units of meters per second squared – m/s^2). In this fashion a mass m has an associated weight $W = m\,g$, where at the Earth's surface $g \approx 9.8\ m/s^2$. Weight is often confused with mass in every day life simply because the gravitational acceleration experienced by people (living, of course, on Earth's surface) is very nearly constant, and consequently the weight of an object doesn't noticeable change as it is moved from one location to another. All this being said, an object of mass m will weigh twice as much in a gravitational field of strength $2g$; it will weigh half as much in a gravitational field of strength $g / 2$.

In the *SI* scheme the mass of an object is measured in kilograms, while weight is measured in the units of Newtons (with $1\ N = 1\ kg\ m / s^2$). Just as with the meter, the determination of a definition and the development of a standard for the kilogram has not gone entirely smoothly. Strictly speaking, the definition for the kilogram runs something like this: a kilogram is the mass of one liter of pure water at standard atmospheric pressure when the temperature is 277.16 Kelvin (3.98^o Centigrade) [3]. The latter temperature condition corresponds to the so-called triple point of water, and is the temperature at which it acquires its greatest density. The actual standard kilogram mass, however, consists of a circular cylinder made of 90% platinum and 10% iridium kept at the *Bureau International des Poids et Measures*, in Paris. The first prototype cylinder was made in 1879 and after careful comparison was deemed to be equivalent in mass to the more formal pure water based definition. The one-kilogram mass cylinder was finally adopted as the standard mass unit at the first General Conference on Weights and Measures held in 1889. The saga of the kilogram definition, however, continues to this very day and recent comparison measurements indicate that the original Paris standard, in spite of careful monitoring and storage, has been loosing mass. In an attempt to remedy this situation a consortium of laboratories from around the world has formed the Avogadro Project[4], which will attempt to redefine the kilogram not in terms of a physical artifact (as it is at the present), but in terms of the equivalent number of carbon-12 atoms.

So, where does all this get us? Well, having agreed to adopt *SI* units we can now describe all physical variables in terms of the base units of mass (m), length (l) and time (s) each raised to some rational power. The standard notation is to write down $[x]$ to denote the units of any physical quantity x with respect to m, l and s. Accordingly, the units for $[x] = l^{\alpha}\, m^{\beta}\, s^{\gamma}$, where α, β and γ are rational numbers. In terms of the three powers (α, β, γ) a physical variable relating to a length measurement will correspond to the numbers $(\alpha, \beta, \gamma) = (1, 0, 0)$; a physical variable relating to a velocity, which is defined as the distance traveled divided by the time taken to travel that distance, will have units described by the powers $(\alpha, \beta, \gamma) = (1, 0, -1)$; a variable relating to a force will be described by the numbers $(\alpha, \beta, \gamma) = (1, 1, -2)$.

To see how all the above discussion helps us with the pendulum, we can derive a formula that links together the period P of the pendulum, being defined as the time to complete one oscillation from the starting point and back again, to the physical quantities associated with a pendulum itself. It does not seem unreasonable to guess that the period should dependent upon $[L]$, the length of the pendulum wire, $[M]$ the mass of the pendulum bob and $[g]$ the gravitational field in which the pendulum swings. What the dimensional analysis approach now says is that the following expression must hold true

$$[P] = k\, [L]^{A}\, [M]^{B}\, [g]^{C} \qquad (1.1)$$

where A, B, C, and k are dimensionless numerical constants. Now, in terms of the units we can write (1.1) as follows

$$\begin{aligned} l^0\, m^0\, s^1 &= [l^1\, m^0\, s^0]^{A}\ [l^0\, m^1\, s^0]^{B}\ [l^1\, m^0\, s^{-2}]^{C} \\ &= [l^{A+C}\, m^{B}\, s^{-2C}] \qquad (1.2) \end{aligned}$$

What we have to do now is make sure that the powers of the base units on both sides of (1.2) are equal – that is, we need to make sure that the units agree. If we equate the powers in length, mass and time separately we have

$0 = A + C$ for the length unit l
$0 = B$ for the mass unit m
$1 = -2C$ for the time unit s

From these relationships we immediately find that B = 0, and consequently derive the result that the period of a pendulum is independent of the mass of its bob. We can also see that C = - ½ and that A = -C = ½. Our dimensional analysis is now complete and we can say that the period of oscillation P for a pendulum of length L in a region in which the acceleration due to gravity is g is

$$P = k\sqrt{L/g} \quad (1.3)$$

where the square root symbol √ has been used to denote powers to one-half. From (1.3) we can deduce, for example, that the period of a pendulum with a wire of length $4L$ will be twice that of a pendulum having a support wire of length L (assuming g remains constant). While equation (1.3) tells us how the period of a pendulum changes with respect to L and g, it does not tell us what the actual period is in seconds. The problem, of course, is that we don't know what the numerical value of the dimensionless constant k is. The value of this term can only be found by a more refined analysis – which will be performed shortly. Interestingly, however, we can be reasonably sure that the value of k won't be an especially large number. No less an authority than Albert Einstein commented upon this latter point in 1911. Specifically, Einstein noted that the dimensionless numbers[5], "are generally of order magnitude one [[6]]. To be sure, this cannot be strictly required, because why should it not be possible for a numerical factor $(12\pi)^3$ to appear in a mathematical-physical analysis? But such cases are unquestionably rare". Indeed, writing upon the unreasonable effectiveness of dimensional analysis in their book *The Anthropic Cosmological Principle* (Oxford University Press, 1986) John Barrow and Frank Tipler suggest that the dimensionless constants are usually small because we live in a low dimensional (that is 3 spatial and one time dimension) universe – we shall pick up on this discussion later. For the moment, however, having argued that the constant in equation (1.3) should be small, what we need to do next is perform the detailed analysis which determines its actual value.

A First Approximate Equation

Sir Isaac Newton entered into our earlier discussion as being the first analyst to use the idea of dimensional analysis – or as he called it, the principle of similitude. It seems appropriate therefore that Newton also provides us with the key mathematical tool needed to determine the constant k in equation (1.3). The mathematical tool is that of calculus[7], which has two basic operations; that of differentiation and that of integration. There will be no attempt to explain the full meaning behind these two operations here, but it is perhaps worth pointing out that the so-called *fundamental theorem of calculus* tells us that the two operations are essentially the inverse of each other. In very general terms differentiation is concerned with determining the rate at which quantities change enabling, for example, a velocity to be described in terms of the time variation of distance traveled. Integration, in contrast, is about adding together infinitely small quantities to find, for example, the total distance traveled in the time interval T_1 to T_2 by an object moving with a variable velocity.

To see how the ideas of differentiation and integration can help us with respect to the pendulum, first take a look at figure 1.1. This figure shows the position of the pendulum at time T_0, when the support wire makes an angle φ_0 with the vertical and the bob is held at rest – that is, the bob's initial velocity $V_0 = 0$ m/s. After some small amount of time Δt, the pendulum bob has moved to a new position for which the support wire makes an angle φ with respect to the vertical. We can now say that in the time interval Δt, the angle describing the position of the pendulum bob with respect to the vertical has changed by an amount $\Delta\varphi = \varphi_0 - \varphi$. What we want to do next is find an expression relating the quantities $\Delta\varphi$, and Δt to the velocity and acceleration of the bob. Before we can do this, however, we must first introduce an additional unit of measure – the radian.

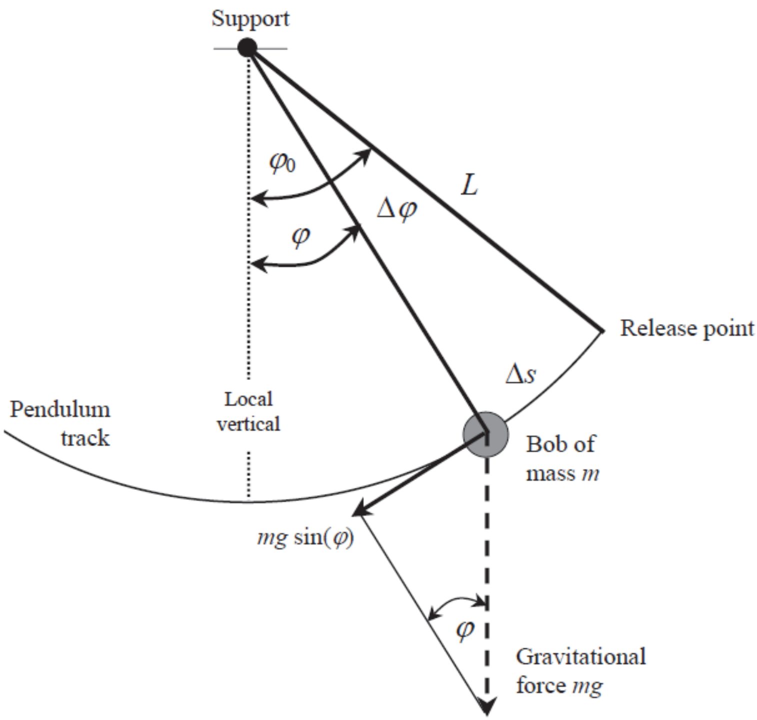

Figure 1.1. The simple pendulum.

The radian is an angular measure, just like the familiar degree unit used in everyday life, but rather than there being 360 degrees subtended in a circle there are 2π radians. The scholars of mathematical history usually tell us that the word 'radian' was first read and witnessed by students sitting an exam set by Professor James Thomas, at Queens College Belfast, on June 5th, 1873. Apparently, however, the idea of the rad, as the radian is commonly abbreviated, had been around much longer and Roger Cotes (1682 – 1716), a student of Sir Isaac Newton's and first Plumian Professor of Astronomy and Experimental Philosophy at Cambridge University in England, is credited with using the concept as early as 1714. By definition one radian is the angle subtended at the center of a circle by an arc along its circumference equal in length to its radius (see figure 1.2). In degree measure, 1 rad = 360 / 2π = $57^{o}.29577951\ldots$, and unless otherwise stated mathematicians always assume that angles are stated in radians. The radian is the 'official' unit of angular measure in the *SI* system, although it is always taken to be a dimensionless number.

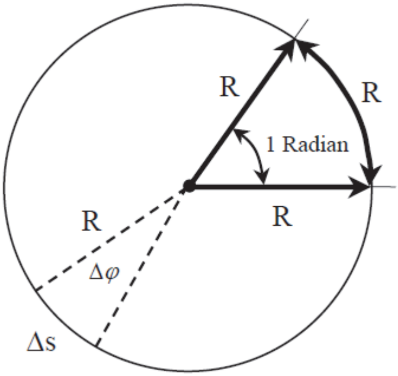

Figure 1.2. The radian is defined as the angle subtended at the center of a circle by a circumference arc equal in length to the circle's radius. One very convenient use of the radian is that the arc length, Δs, subtended by an angle $\Delta\varphi$ can be expressed as the product: $\Delta s = \Delta\varphi\, R$, where R is the radius of the circle.

With the radian now defined the distance traveled along the arc of the pendulum track Δs (see figure 1.1) can be written as $\Delta s = L\,\Delta\varphi$, and the velocity of the pendulum bob is accordingly $V = \Delta s \,/\, \Delta t = L\,\Delta\varphi \,/\, \Delta t$. The acceleration, defined as the rate of change of the velocity, experienced by the pendulum bob in time Δt can further be expressed as $a = \Delta V \,/\, \Delta \mathrm{t} = (V - V_0) \,/\, \Delta t = L\,(\Delta\varphi \,/\, \Delta t) \,/\, \Delta t$. Now, the key point about differential calculus is that it is concerned with variations when the Δ terms approach the limit of becoming infinitely small. In this manner the differential expressions for the velocity and the acceleration are respectively $V = L\,(d\varphi \,/\, dt)$ and $a = dV \,/\, dt = L\,(d^2\varphi \,/\, dt^2)$ – the *d* symbol indicates the infinitely small limit of the Δ's have been taken. In words, what these two equations tell us is that the velocity is related to the first time derivative of the position angle φ, and that the acceleration is related to the second time derivative of the position angle φ. The velocity is therefore described according to the rate at which the angle φ

is swept out, and the acceleration is expressed as the rate of change at which the angle φ is swept out.

Having derived expressions for the velocity and acceleration of the pendulum bob, we now want to relate these to the gravitational force *mg* acting upon the bob. Here recall, *m* is the mass of the bob in kilograms and *g* is the acceleration due to gravity. The product *mg* is the gravitational force, acting downwards, between the pendulum bob and the Earth. We shall have much more to say about gravity and gravitational forces later on, but for the moment it can be seen from figure 1.1 that the component of the force acting to move the bob along its constrained arc has a magnitude $mg \sin(\varphi)$. It is this component of the gravitational force that initially acts to accelerate the pendulum bob away from its release point.

To complete the derivation we must now use Newton's second law of motion, which relates acceleration to an applied force. Specifically, what Newton's second law tells us is that if a force F acts upon an object of mass m, then the acceleration a produced is given by the expression: $a = F / m$. For our pendulum the force $F = -mg \sin(\varphi)$ where the minus sign has been introduced by convention since gravity acts in a downwards direction. We have already seen, however, that the acceleration is related to the second derivative of the angle φ, and we can accordingly write: $L\, d^2\varphi/dt^2 = a = F/m = -g\sin(\varphi)$. Now, this equation is all well and good, but we still haven't found our sought after expression for the pendulum period. To make progress in this direction we must first make use of a convenient simplification related to radians. Specifically, it turns out, the value of the quantity $\Psi = \sin(\varphi) / \varphi$, when φ is expressed in radians, approaches unity as φ goes to zero. Schematically we can write this as $\Psi \to 1$ as $\varphi \to 0$. Where this approximation helps us is that the differential equation for the pendulum motion can, when φ is small, be written as $L\, d^2\varphi / dt^2 = -g [\sin(\varphi) / \varphi]\, \varphi \approx -g\, \varphi$. The ≈ sign has been used in our new expression to indicate that the approximation only holds true if the angle φ through which the pendulum swings is small. The reason why this small angle approximation helps us is that we can now find an exact solution to our differential equation by substitution. Specifically, the solution follows by writing $\varphi(t)$, the angle φ at time t after the release of the pendulum bob, as $\varphi(t) = \varphi_0 \cos[\,(2\pi / P)\, t\,]$ where φ_0 is the initial angle at the time of release (assumed small remember), and P is the time for the pendulum to complete one oscillation. The motion is constrained to move between φ_0 and $-\varphi_0$, with the lowest point on the pendulum's track being the vertical position $\varphi = 0$ (corresponding to an instantaneous plumb-line). If we now differentiate the expression for $\varphi(t)$ with respect to time t twice and substitute the resultant terms into the differential equation for the pendulum motion[8] then the expression for the period P that we have been looking for emerges, namely, $L(2\pi / P)^2 = g$, which is more conveniently written as

$$P = 2\pi\sqrt{L/g} \qquad (1.4)$$

So, now we have our answer, a complete, first approximation result for the period of oscillation to a simple, small angle of oscillation, pendulum. Comparing equation (1.4) with equation (1.3) we find, just as the dimensional analysis predicted, the period varies according to the square root of the length of the pendulum L divided by the acceleration due to gravity g. But even more usefully, we also find that the dimensionless constant $k = 2\pi = 6.283185307...$ (a relatively small number, of order ten, as Einstein argued should be the case). This result is helpful since we can now calculate the actual period of a pendulum in seconds[9].

While equation (1.4) describes a useful relationship between the period of a pendulum, its length and the gravitational field strength[10], it is interesting to note what quantities the period doesn't depend upon. From our dimensional analysis, for example, the period was found to be independent of the mass of the bob, and from the first approximation analysis it is also found that the period is independent of the initial angle (φ_0) that it is released from (well, up to a point, as we shall see below). These are remarkable observations, and they underscore the great importance of the pendulum with respect to time keeping. For a given length of support wire the period is isochronal: the swing of a pendulum literally slices time into equal intervals irrespective of its starting conditions. In addition, it is also found that it is of no great consequence what the pendulum bob is made of; clay, wood, gold or lead it is irrelevant to the pendulum's period of swing. This latter point is, in fact, of great fundamental interest, as Isaac Newton first realized and as we shall see later in Chapter 4.

The Seconds Pendulum

The standard unit of time measurement is the second; a word derived from the ancient Greeks who called one sixtieth of a degree the 'first small part', and one sixtieth of 'the first small part' was, appropriately enough, called the 'second small part'. In Latin the 'first small part' would be written as *pars minuta prima* while the 'second small part' would be written as *pars minuta secunda*, and it is from these expressions that the words minute and second have descended. The fact that the divisions are taken as the $1/60^{th}$ part is due to the ancient Babylonians, who used 60 as the base of their counting scheme (the sexagenary system) – as a side note, it was the ancient Egyptians who introduced the idea of dividing the day into 24 hours.

Since the period of the small-amplitude pendulum is constant (that is, isochronal) and independent of its starting position, it seems only natural to ask, what is the length of a pendulum that takes one second to complete one swing – that is, what is the length of a pendulum that has a period of two seconds. The point being, of course, that once calibrated such a pendulum could provide a standard for time that is easily transportable from one location to another. Not only this, the length of the Seconds Pendulum could also be adopted as a standard unit of length. Indeed, it was originally intended that the meter should be based upon the length of the Seconds Pendulum as determined at the latitude of 45 degrees near the town of Bordeaux in France. This specific proposal was first presented to the new Revolutionary Government of France by Charles-Maurice de Talleyr and in 1790, after some no doubt complicated but certainly skilful political wrangling, both Britain and the newly confederated United States of America agreed to adopt the proposed unit of length. The Commission of Weights and Measures to the French Government met, however, on March 19th 1791, and, rather surprisingly, decided to drop the idea of the Seconds Pendulum as the bases for the standard of measure, proposing instead the one ten millionth distance from the North Pole to the Equator. Neither Britain nor the United States were particularly impressed with the new scheme and accordingly declined to join in the geodetic survey to map out the meridian from which the meter would be derived. The reason why the length of the Seconds Pendulum was dropped from consideration as the standard of measure was explained by the Committee along the lines that it would be based upon the unit of time (i.e., the second), and this, they argued might change. Indeed, the Revolutionary Government at that time was seriously considering the installation of a decimalized day of 10 hours duration, with each hour being 100 minutes long and each minute being composed of 100 seconds[11]. These new hours, minutes and seconds, of course, would be of different duration to our Babylonian based measures of the same name. Additionally, the Commission argued that by basing the meter on the Earth's quarter circumference it would reflect the global nature of the new standard and would apply to all humanity. Critics of the new scheme immediately pointed out (but to no avail) that the new definition was itself entirely arbitrary and was not even a real distance, but one based upon a mathematical calculation of an imaginary arc extrapolated from a small measured segment of the whole. Well, of course, no one said that agreeing on fundamental units was going to be easy. The important point of this story, however, is that as a result of its decision the French Government set about sponsoring several scientific expeditions to carefully measure the Earth, and pendulum based experiments were an important part of the survey reductions - as we shall see later in Chapter 3.

The length of a seconds pendulum can de determined with the aid of equation (1.4). The calculation proceeds by setting the period P equal to 2 seconds, and taking g to be the standard reference acceleration due to gravity as 9.806650 m/s^2. Accordingly, the length of the standard Seconds Pendulum is 0.993621 meters.