The Infinite Universe of Einstein and Newton

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Chapter 1

The Infinite Universe

This chapter presents physical ideas grounded in the observed isotropic homogeneous nature of the Universe that lead to the conclusion that the Universe might be infinite. Following this possibility, three metrics for an infinite, static, isotropic, and homogenous universe are found, which satisfy the field equations of Einstein's general relativity. Two of these metrics show that light traveling in their universe would be gravitationally red shifted. In addition, the universe of one of these two metrics, the infinite closed universe, would trap light in a finite portion of the universe, so that each point in space would have its own finite universe. The second chapter concludes by showing that the infinite closed universe fits the data of the Hubble diagram better than the Big Bang Theory.

1.1 Physical Ideas

There are three fundamental observational facts about the nature of our Universe. One is that the Universe on a very large scale seems to be homogeneous and isotropic. The second observation is that the light coming from far away objects is red shifted. The further the object is away from us, the larger on average is the redshift. The third fact is that the night sky is dark. This last observation presents a problem—known as Olbers' paradox: "If the Universe is infinite with a finite number of stars per unit volume then the night sky would be expected to have a brightness equal to the brightness of the surface of an average star."

The currently held view is that the Universe is finite and expanding everywhere and that this expansion is the cause of the cosmological redshift. Every one acknowledges the existence of gravitational red shifting. However, it is thought that gravitational red shifting effects are much too small to account for the observed cosmological redshift. But are they? The Schwarzschild line element (Adler et al. 1975, d'Inverno 1992, Weinberg 1972) for a spherically symmetric static massive "object" exterior to the object is

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{1}{1 - \frac{2GM}{c^2 r}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \tag{1.1}$$

The mass M for any spherical object of constant density ρ is $M = \frac{4}{3}\pi\rho r^3$. So that when $r = \sqrt{\frac{3c^2}{8\pi G\rho}}$, g_{rr} in the Schwarzschild metric becomes infinite and g_{tt} equals zero. This means that no matter how small the density, there is a radius for an object of that density for which it would be a black hole. Consequently, light emanating from near the object's center would be severely redshifted as it approached the boundary of the object.

How big would our Universe be if it were a black hole so that no light could pass out of it? If we assume a density of $\rho = 1 \times 10^{-27} \mathrm{kg\,m^{-3}}$, which is about half the currently accepted best value, the Schwarzschild radius of our Universe would be about 42 billion light years. This is about 1.4 times the presently accepted value of the radius of the universe "now". If one used the current best value of the density of the universe one would get a Schwarzschild radius for our Universe about equal to the currently accepted value for the radius of our Universe. However, this is not the point. Whatever else this tells us, it is clear that gravitational shifting need not be negligible and that any light from a very distant source would in general have its wavelength severely shifted due to gravity.

Light traveling through a finite diffusely filled universe, no matter how large, is similar to a ball traveling through a smooth frictionless tunnel carved inside the Earth. A ball in such a tunnel will pick up speed (kinetic energy) as it comes closer to the center of the Earth and it will slow down, that is, lose kinetic energy as it gets farther from the center. Similarly, light would lose energy, that is, its frequency, $f = E/h$, would decrease, as the light gets further from the center of a finite universe. Conversely, the frequency of light would increase as light gets closer to the center of such a universe. Put in terms of wavelength, if the source of the light were closer to the center of the universe than an observer, the observer would observe the light as red shifted. However, if the source of the light were further from the center of the universe than the observer then the light would appear blue shifted. If our Universe were an expanding finite universe, the light would be redshifted because of the expansion. However, because of the size of the gravitational shifting of light, there would be a clear and undeniable asymmetry in the observed redshift for any observer not situated at the center of the Universe. Since this is not what we observe, this suggests that the Universe may be infinite. The Big Bang Universe assumes a different nature for the geometry of a finite universe in order to circumvent this problem. We, however, will pursue the possibility of an infinite universe with ordinary type geometry.

1.1.1 The Infinite Limit Conception

As I was taught and as one teaches in freshman physics, the gravitational force on a material particle in a region which has a finite spherically symmetric mass density, no matter how large the sphere, is toward the center of the sphere and has a magnitude that is only dependent on the matter which is closer to the center than the particle. Further, the net force on the particle depends not a whit on the amount of matter which is farther from the center than it. In addition, the potential energy between the particle and mass distribution increases as the particle gets further and further from the center. If one considers an isotropic homogeneous infinite universe with uniform non-zero density to be the limit of a finite spherical universe, which keeps its density fixed as $r \to \infty$, then this potential energy property would hold for the infinite universe too.

In a homogeneous, isotropic, infinite universe each point in space is equivalent to any other point on the cosmological scale. That is, each point in space can be considered an origin about which gravitating "matter" is spherically symmetric in the cosmological sense. Then, in the infinite limit conception, as a light ray "pulls away" from its source, the origin of its coordinates, which corresponds to the same natural origin of a spherical wave, more and more gravitating mass lies interior to it. This means the light's potential energy rises and thus its frequency, ie., its energy decreases. So, in an isotropic, homogeneous, infinite universe, light is reddened the further it gets from its originating source. In addition, since the universe is homogeneous, it is reddened on average by the same amount for all light which has traveled the same distance. This means an observer will see light which has been red shifted coming at him from all directions. The light will be red shifted more, the further away the source. Furthermore, the isotropy of such an infinite universe would mean that the red shifting of observed light would be, in the large, independent of direction.

This conception of light pulling away from its source point and "seeing" more and more gravitating mass in an homogeneous infinite universe gives impetus and focus to what is to follow. However, this conception connected with infinite space might seem somewhat counter-intuitive because one might expect the "pull" from one side to cancel the "pull" from the opposite side, and thus there might be no net effect and the space would be flat. Nevertheless, we proceed. (It turns out that later we will see a resolution to these conflicting symmetries.)

1.1.2 Classical Analysis

In this section we will analyze the static homogeneous, isotropic, infinite universe classically, using the infinite limit conception, in order to test whether it matches our qualitative reasoning above. To do this we consider two concentric spherically symmetric regions of constant density ρ_1 and ρ_2 as shown in Fig. 1.1.

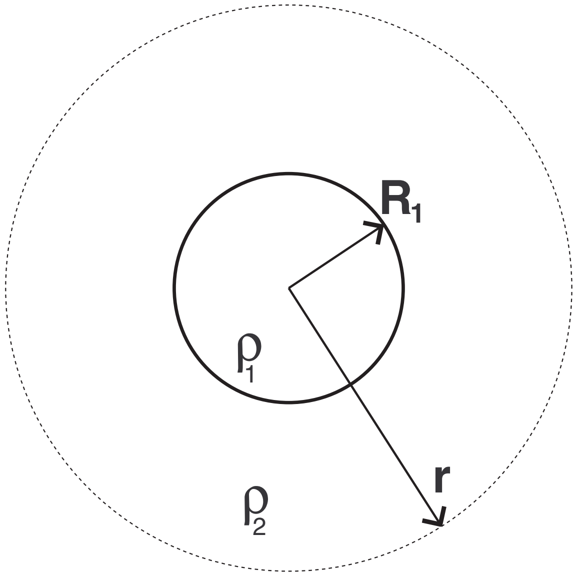

Figure 1.1: Two concentric spherically symmetric regions used to help define variables.

In region 1, the force, $\mathbf{F_1}$ on a particle of mass, m, is equal to $-GM_{in}m/r^2\hat{\mathbf{r}}$ where $M_{in} = \frac{4}{3}\pi\rho_1 r^3$. Since $\mathbf{F_1} = -\boldsymbol{\nabla} U_1$, the potential energy associated with the particle in region 1 is given by

$$U_1 = \frac{2\pi\rho_1 G m r^2}{3} + k_1.$$

Similarly, for a particle of mass, m, in region 2 the potential energy is given by

$$U_2 = -\frac{\frac{4\pi}{3}R_1^3(\rho_1 - \rho_2)Gm}{r} + \frac{2\pi}{3}G\rho_2 m r^2 + k_2.$$

For a finite universe of radius R_1 ($\rho_2 = 0$) one would set $k_2 = 0$, so that the gravitational potential energy is zero at infinity. However, for an infinite universe one takes the gravitational potential energy, zero reference level at the origin by setting $k_1 = 0$. Then, requiring continuity of the potential energy at R_1 gives $k_2 = 2\pi Gm(\rho_1 - \rho_2)R_1^2$, which results in

$$U_1 = \frac{2\pi}{3}Gm\rho_1 r^2 \tag{1.2a}$$

and

$$U_2 = 2\pi Gm(\rho_1 - \rho_2)R_1^2 - \frac{Gm\frac{4\pi}{3}(\rho_1 - \rho_2)R_1^3}{r} + \frac{2\pi}{3}Gm\rho_2 r^2. \tag{1.2b}$$

For the infinite universe, let us consider the simple idealized case in which light is emitted from some atom in space (the origin) and not from the surface of a star. To do this we set $\rho_1 = \rho_2 = \rho$. Then, the potential energy is

$$U = \frac{2\pi}{3}Gm\rho r^2 \tag{1.3a}$$

or for later use, the potential is given by

$$V = \frac{2\pi G\rho r^2}{3}. \tag{1.3b}$$

We apply conservation of energy, utilizing Eq. (1.3a), and set the energy of the light at the source equal to the energy of the light at the observer. This gives

$$hf_0 = hf + \frac{2\pi}{3}G\Big(\frac{hf}{c^2}\Big)\rho r^2, \tag{1.4}$$

where $\frac{hf}{c^2}$ has been used for the "mass" of the photon. This means that classically, the frequency of the light seen by the observer, in terms of the frequency of the light emitted by the source, is

$$f = \frac{f_0}{1 + \dfrac{2\pi G\rho}{3c^2}r^2}.$$

In other words, the wavelength of the observed light is reddened with an observed wavelength given by

$$\lambda = \Big(1 + \frac{2\pi G\rho}{3c^2}r^2\Big)\lambda_0. \tag{1.5}$$

Note, that in this analysis the light is not trapped in a black hole since the energy of a photon is only reduced to zero in the limit as r approaches infinity. Thus, this analysis is not fully consistent with general relativity, since no black hole is manifest. However, the analysis does show that light is red shifted as it moves away from its source.

If the light leaves the surface of a star of radius R_1 then conservation of energy results in the equation

$$\begin{aligned} hf_0 + \frac{2\pi}{3}G\Big(\frac{hf_0}{c^2}\Big)\rho_{st}R_1^2 \quad = \quad & hf + \frac{2\pi}{3}G\Big(\frac{hf}{c^2}\Big)\rho_{sp}r^2 \\ & - \frac{G(\frac{hf}{c^2})\frac{4\pi}{3}(\rho_{st} - \rho_{sp})R_1^3}{r} \\ & + 2\pi G\Big(\frac{hf}{c^2}\Big)(\rho_{st} - \rho_{sp})R_1^2. \end{aligned}$$

Assuming that the density of space ρ_{sp} is negligible compared to ρ_{st} of the star and that the observer is far from the star so that the $1/r$ term can be neglected, the expression for conservation of energy reduces to

$$hf_0 + \frac{2\pi}{3}G\Big(\frac{hf_0}{c^2}\Big)\rho_{st}R_1^2 = hf + \frac{2\pi}{3}G\Big(\frac{hf}{c^2}\Big)\rho_{sp}r^2 + 2\pi G\Big(\frac{hf}{c^2}\Big)\rho_{st}R_1^2.$$

Thus, the frequency of the observed light is given by

$$f = \frac{1 + \dfrac{2\pi G\rho_{st}R_1^2}{3c^2}}{1 + \dfrac{2\pi G\rho_{st}R_1^2}{c^2} + \dfrac{2\pi}{3c^2}G\rho_{sp}r^2} f_0.$$

Here, we see that light from a star is red shifted similar to light emanating from empty space. However, as before we see that no black hole is manifest, no matter how massive the star. Since, recent observations have made it seem very likely that there are black holes at the center of many galaxies (Genzel et al. 1997, Ghez et al. 1998) and have also indicated that there are black hole double "stars" in which one of the massive objects is a black hole (Casares and Charles 1994, Shahbaz et al. 1994), the existence of black holes is all but confirmed. Thus, the results above indicate the need to do a full general relativistic analysis whenever the gravitational effects are very large.

1.2 General Relativistic Analysis

In order to proceed with the general relativistic analysis, we will take a metric of a most general form which satisfies the conditions which are observed and also satisfies the infinite universe hypothesis. We want our metric to represent a universe which is homogeneous, isotropic and static in the cosmological sense about any point in the Universe. In order to isolate the cosmological properties of the Universe from the properties due to local irregularities, we will consider that all the matter in the Universe is spread out into a uniform matter density.

Because we are considering the universe to be static, time becomes irrelevant, that is, all times are equivalent. This is so because in a static universe, from a given position in space, the universe would always present similar physics in the cosmological sense no matter when observations are made. Thus, the homogeneity we are assuming allows observations by various observers, which can be made at any time, and only requires that observations of different observers "look" the same in the cosmological sense. Thus, it is not necessary that all observers be able to coordinate their times. So, in order to keep our assumptions to a minimum, we do not make the common cosmological assumption concerning the existence of a global Gaussian coordinate system. In particular, we do *not* assume that there exists a global time coordinate, $x^0 = ct$, such that the Gaussian metric

$$ds^2 = c^2 dt^2 + g_{ij} dx^i dx^j \qquad i, j = 1, 2, 3 \tag{1.6}$$

exists globally. Instead, we let the nature of time come out as it will. In this way we let the cosmological properties of our model be the result of its matter distribution and gravitation alone.

The development of the Schwarzschild metric (Adler et al. 1975, d'Inverno 1992, Weinberg 1972) begins with writing a canonical general form for a metric which is static and isotropic about a particular point—the origin. The origin might be the center of a star or black hole. Or, it might be nothing at all. The general metric chosen in the Schwarzschild development is defined by the line element

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \tag{1.7}$$

That is, the metric is

$$(g_{ab}) = \begin{pmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}. \tag{1.8}$$

We choose the same general metric that is chosen for the Schwarzschild metric as our starting point except, that for us, the origin is not a fixed particular point but is a fixed arbitrary point. We do this because each point in a static, isotropic, homogeneous, infinite universe is equivalent. That is, the origin can be taken to be at any point in space-time. This will guarantee that any metric solution we find is spatially homogeneous because the solution will be isotropic about every point in space-time by assumption.

We will proceed by solving Einstein's field equations for general relativity,

$$G_{ab} = \kappa T_{ab} - \Lambda g_{ab} \tag{1.9}$$

where the Einstein tensor G_{ab} is given by

$$G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R. \tag{1.10}$$

For completeness of definitions:

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$$

where $R_{bd} = R^a{}_{bad}$, and

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

The Schwarzschild problem consists of a spherically symmetric massive object centered at the origin. Beyond the object is empty space. This means beyond the object, the matter energy-momentum tensor, T_{ab}, in the Schwarzschild case, is identically zero. The Schwarzschild solution, exterior to the object, is obtained by solving the Einstein field equations, $G_{ab} = 0$. Note these are simplified field equations since they assume $\Lambda = 0$. The solution of these equations is the line element of the Schwarzschild solution, Eq. (1.1). Far from the spherically symmetric object the form of the line element is

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \tag{1.11}$$

We note, this is the flat-space Lorentz line element of special relativity written in spherical polar coordinates. Note too, if the massive body were not, that is, if its mass were zero then again the exterior Schwarzschild solution line element, Eq. (1.1), would be the homogeneous flat-space Lorentz line element Eq. (1.11), everywhere. That is, the metric line element for an infinite empty universe is the homogeneous Lorentz metric of Eq. (1.11).

In order to see the results of the infinite universe hypothesis without complicating factors, we will consider the idealized case of light emitted from empty space in a static, homogeneous, isotropic, diffusely filled infinite universe. This will be essentially the same problem as the Schwarzschild problem without a mass, M, at the origin, except that the matter energy-momentum tensor, $T^a{}_b$, will not be zero. Our hypothesis assumes a static, homogeneous, isotropic matter-energy distribution throughout the Universe. That is, we are assuming on average, on the cosmological scale that $v/c \ll 1$. In this case, the matter energy-momentum tensor is

$$(T^a{}_b) = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p(r) & 0 & 0 \\ 0 & 0 & -p(r) & 0 \\ 0 & 0 & 0 & -p(r) \end{pmatrix}$$

where ρ is the matter-energy density (mass density) of the Universe and $p(r)$ is a "pressure" (Adler et al. 1975). We include both matter and energy (E/c^2) in mass density.

Our usual experience with finite bodies is that pressure resists volume reduction. For example, the reaction of a fluid to resist compression or the radiation pressure within a star both of which, stabilize the body against collapse. Here, however, since we are assuming a static universe, the "pressure" comes from gravity itself. Because we are assuming a uniform, isotropic density ρ, each matter point in space can be expected to be pulled on by its neighbors. However, we also choose a reference point to which we anchor an origin of coordinates. When we do this, we have two competing effects: the attraction—"pressure"—toward the origin caused by the spherically symmetric distribution of matter closer to the origin than the point under consideration and the pull in other directions—"pressure"—of all the other matter in the Universe within range. Upon some reflection, due to the different range and symmetry possibilities we might assume for the pressure, one might even expect three solutions to Einstein's field equations will be possible. Each one representing one of the three basic geometries; open, closed and flat. Reality might then be expected to tell us which, if any, is correct.

We need to solve Einstein's field equations

$$G^a{}_b = \kappa T^a{}_b - \Lambda g^a{}_b. \tag{1.12}$$

where Λ is the cosmological constant. Without the cosmological constant, these equations tell us that the gravitational field represented on the lefthand side of the field equations by $G^a{}_b$ is coupled to matter represented on the righthand side of the equations by $T^a{}_b$ through a coupling constant κ. Einstein introduced the cosmological constant in order to solve the very problem we are attempting; namely, a universe with constant density (Einstein 1917, 1995, 1996). Einstein found that in order to solve for a finite static universe, it seemed necessary to introduce either a negative pressure or the cosmological constant. Einstein chose the constant as the lesser of two evils. He was unhappy with the constant's introduction since it complicated the logical simplicity of the theory (Einstein 1996, 1995). However, we will include it in our development because it preserves the full generality of the gravitational field equations for $g_{\mu\nu}$. We take the point of view in our work that we will not assume more than we have to. That is, we make no assumption in order to simplify the fully general field equations. We will instead try to let nature decide between possibilities. It turns out we could leave the value of the coupling constant, κ, undetermined in our work. Its theoretical value is $\kappa = 8\pi G/c^4$.

There are only four non-trivial Einstein field equations. Noting that $g^a{}_b = \delta^a_b$, the non-trivial field equations are: $G^0{}_0 = \kappa\rho c^2 - \Lambda, G^1{}_1 = -\kappa p(r) - \Lambda$, $G^2{}_2 = -\kappa p(r) - \Lambda$, and $G^3{}_3 = -\kappa p(r) - \Lambda$. Only the first three equations are independent because $G^3{}_3 = G^2{}_2$. Written in order, letting $\Omega = -\kappa p(r) - \Lambda$, these equations are:

$$\frac{1}{r^2} - \frac{e^{-\lambda(r)}}{r^2} + \frac{e^{-\lambda(r)}\lambda'(r)}{r} = \kappa\rho c^2 - \Lambda \tag{1.13a}$$

$$\frac{1}{r^2} - \frac{e^{-\lambda(r)}}{r^2} - \frac{e^{-\lambda(r)}\nu'(r)}{r} = \Omega \tag{1.13b}$$

$$e^{-\lambda(r)}\left(\frac{\lambda'(r)}{2r} - \frac{\nu'(r)}{2r} + \frac{\lambda'(r)\nu'(r)}{4} - \frac{\nu'^2(r)}{4} - \frac{\nu''(r)}{2}\right) = \Omega. \tag{1.13c}$$

These equations are developed in exactly the same manner as the equations in the Schwarzschild problem. Readers may easily check these equations with those developed in d'Inverno's text (d'Inverno 1992) on page 187, since he has used the same sign conventions.

The first equation, Eq. (1.13a), can be rewritten

$$\begin{aligned} 1 - \left(\kappa\rho c^2 - \Lambda\right) r^2 &= e^{-\lambda} - \lambda' r e^{-\lambda} \\ &= \left(r e^{-\lambda}\right)'. \end{aligned} \tag{1.14}$$

Thus we have

$$e^{-\lambda} = 1 - \frac{\kappa\rho c^2 - \Lambda}{3} r^2 + \frac{k_1}{r} = 1 - \alpha r^2 + \frac{k_1}{r} \tag{1.15}$$

where $\alpha = \frac{1}{3}\left(\kappa\rho c^2 - \Lambda\right)$ and k_1 is a constant of integration. In the Schwarzschild problem, k_1 of the $1/r$ term is proportional to the central mass. However, the r^2 term, which comes about due the non-zero density of "free" space, is absent in the Schwarzschild problem since it assumes free space is empty, ie., $\rho = 0$. Because, we are assuming the idealized cosmological model in which light is emanating from $\rho \neq 0$ free space and not from the surface of a star, we take $k_1 = 0$. That is, we assume there is no central mass at the origin of the coordinates. Thus, we have

$$g_{11} = -e^{\lambda} = \frac{-1}{1 - \alpha r^2} \tag{1.16}$$

and

$$\frac{d\lambda}{dr} = \frac{2\alpha r}{1 - \alpha r^2}. \tag{1.17}$$

Note that equation (1.13a) is independent of $\nu(r)$ so that any solution we may find for $\nu(r)$ will be consistent with equation (1.13a). In addition, note that both the left hand side of equation (1.13b) and the left hand side of equation (1.13c) are equal to $\Omega(r)$. If we set the left hand sides of equations (1.13b) and (1.13c) equal, we will get a single equation consistent with both. If into this equation we plug in our solution for $\lambda(r)$ and solve the resulting equation for $\nu(r)$, we will have solutions, $\nu(r)$ and $\lambda(r)$ consistent with the complete set of field equations.

The equation which results from the above procedure when we set $k_1 = 0$ in Eq. (1.15) is

$$\frac{1}{r^2} - \frac{1-\alpha r^2}{r^2} - \frac{1-\alpha r^2}{r}\nu' =$$
$$\left[1-\alpha r^2\right]\left[\frac{\alpha}{1-\alpha r^2} - \frac{\nu'}{2r} + \frac{\alpha r \nu'}{2(1-\alpha r^2)} - \frac{\nu'^2}{4} - \frac{\nu''}{2}\right]$$

which reduces to

$$\frac{\nu'}{r(1-\alpha r^2)} - \frac{\nu'^2}{2} - \nu'' = 0. \tag{1.18}$$

A method which can be used to find the solution to Eq. (1.18) is to reduce it to a first order differential equation by using the substitution $p = \nu'$. This gives the first order equation

$$p' - \frac{1}{r(1-\alpha r^2)}p = -\frac{p^2}{2}, \tag{1.19}$$

which is a Bernoulli equation. This particular Bernoulli equation can be reduced to a linear first order differential equation with the substitution $y = p^{-1}$. This substitution gives rise to the equation

$$y' + \frac{1}{r(1-\alpha r^2)}y = \frac{1}{2} \tag{1.20}$$

which produces the solution

$$y = \frac{2\alpha k_2\sqrt{1-\alpha r^2} - (1-\alpha r^2)}{2\alpha r} \tag{1.21}$$

where k_2 is a constant of integration. That is, we have

$$\nu'(r) = \frac{2\alpha r}{2\alpha k_2\sqrt{1-\alpha r^2} - (1-\alpha r^2)}. \tag{1.22}$$

A straight forward integration produces the solution

$$\nu(r) = \ln\left(\sqrt{1-\alpha r^2} - 2\alpha k_2\right)^2 + k_3. \tag{1.23}$$

So that we have the solution

$$\begin{aligned} g_{00} &= e^{\nu(r)} \\ &= e^{k_3}\Big[1 - \alpha r^2 + 4\alpha k_2 \left(\alpha k_2 - \sqrt{1-\alpha r^2}\right)\Big], \end{aligned} \tag{1.24}$$

where α is to be determined.

To this point, we have solved Einstein's equations for the general metric line element of Eq. (1.7) for our idealized cosmological model. We have obtained expressions for g_{00} and g_{11} of that general line element in Eq. (1.24) and Eq. (1.16) in terms of three constants. In order to determine the particular solutions to our infinite universe model, if any, we need to apply the boundary conditions for our model.

1.2.1 Application of Boundary Conditions

Standard analysis proceeds by writing the metric, g_{ab} as the series expansion $g_{ab} = \eta_{ab} + \epsilon h^{(1)}_{ab} + \epsilon^2 h^{(2)}_{ab} + O(\epsilon^3)$ where the lead term, η_{ab} is the Lorentz flat space metric. In our simplified model, flat spacetime occurs at the origin. In order to simplify the presentation, in our expression for g_{00} of Eq. (1.24), we set e^{k_3} equal to one. (It turns out that if we were to apply the boundary conditions with arbitrary k_3 we would obtain no new solutions.) That is, we set k_3, the constant of integration in Eq. (1.23), equal to zero. Now, expanding the expression for g_{00} given in Eq. (1.24), with $k_3 = 0$, as a power series in the radius r about the origin, we find

$$\begin{aligned} g_{00} &= \left[1 + 4k_2\alpha\left(k_2\alpha - 1\right)\right] + \left(2k_2\alpha^2 - \alpha\right) r^2 \\ &\quad + \frac{1}{2}k_2\alpha^3 r^4 + \frac{1}{4}k_2\alpha^4 r^6 + \cdots. \end{aligned} \tag{1.25}$$

In order that we have Lorentz flat-space at the origin, the lead term in this series equals 1. Thus, we must require either $k_2 = 0$, $\alpha = 0$, or $k_2 = 1/\alpha$.

$k_2 = 0$

As is easily seen from Eq. (1.24), if k_2 equals zero, the expression for g_{00} given in Eq. (1.25) becomes

$$g_{00} = 1 - \alpha r^2. \tag{1.26}$$

Standard analysis shows that

$$g_{00} = 1 + \frac{2V}{c^2} \tag{1.27}$$

in the Newtonian limit of weak fields and low velocity, where V is the gravitational potential (Adler et al. 1975, d'Inverno 1992, Weinberg 1972, Misner et al. 1973). Substituting for g_{00} and V in Eq. (1.27) using Eq. (1.26) and Eq. (1.3b), we get

$$\alpha = -\frac{4\pi G\rho}{3c^2}, \tag{1.28}$$

or since $\alpha = \frac{1}{3}\left(\kappa\rho c^2 - \Lambda\right)$ and $\kappa = 8\pi G/c^4$

$$\Lambda = \frac{12\pi G\rho}{c^2}. \tag{1.29}$$

When we substitute the results of Eqs. (1.26), (1.16), and (1.28) into the metric line element of Eq. (1.7), we get

$$\begin{aligned} ds^2 \quad = \quad & \left(1 + \frac{4\pi G\rho}{3c^2} r^2\right) c^2 dt^2 - \frac{1}{1 + \dfrac{4\pi G\rho}{3c^2} r^2} dr^2 \\ & - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \end{aligned} \tag{1.30}$$

as the line element for a static, diffusely matter filled, homogeneous, isotropic, infinite universe. There is no discontinuity in this metric line element in the r coordinate because g_{rr} remains negative and finite everywhere. In addition, g_{00} remains positive everywhere, with $g_{00} \to \infty$ as $r \to \infty$. This suggests that such a universe remains causally and visually connected to infinity. Since, among other things, this would not to lead to a clear resolution of Olbers' paradox, we tentatively discard this solution as a model for our universe.

$\boldsymbol{\alpha = 0}$

If $\alpha = 0$, with α defined in Eq. (1.15), the cosmological constant Λ equals $\kappa\rho c^2$. Using Eq. (1.24), we find $g_{00} = 1$. Further, by Eq. (1.7) and Eq. (1.16), we have $g_{rr} = -e^{\lambda} = -1$. Thus, the $\alpha = 0$ solution is the flat-space Lorentz metric line element,

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{1.31}$$

of special relativity. This solution will not satisfy the boundary condition for the r^2 term in Eq. (1.25) using the potential of Eq. (1.3b), which is compatible with the spherically symmetric infinite limit conception. However, as we noted earlier, the acceptance of the spherically symmetric boundary conditions might seem counter-intuitive. Instead, consistent with our expectation that the "pull" from one side might cancel the "pull" from the other so that there would be no net effect due to a uniform distribution of matter, we may assume that the potential is constant. The $\alpha = 0$ solution of Eq. (1.31) is consistent with such a boundary condition. Nevertheless, since the solution is causally and visually connected to infinity, it does not provide an apparent resolution to Olbers' paradox, and so we tentatively discard it as a model for our universe.

$\boldsymbol{k_2 = 1/\alpha}$

We analyze the $k_2 = 1/\alpha$ solution in the same manner that we analyzed the $k_2 = 0$ solution. If $k_2 = 1/\alpha$ then, expanding Eq. (1.24),

g_{00} becomes

$$\begin{aligned} g_{00} &= 1 - \alpha r^2 + 4\left[1 - \sqrt{1 - \alpha r^2}\right] \\ &= 1 + \alpha r^2 + \frac{1}{2}\alpha^2 r^4 + \frac{1}{4}\alpha^3 r^6 + \cdots . \end{aligned} \tag{1.32}$$

Plugging the lowest order terms of g_{00} into Eq. (1.27) along with the value of the potential given in Eq. (1.3b), we get

$$\alpha = \frac{4\pi G\rho}{3c^2} \tag{1.33}$$

or

$$\Lambda = \frac{4\pi G\rho}{c^2}. \tag{1.34}$$

Note from Eq. (1.32), we can write g_{00} as

$$g_{00} = \left[2 - \left(1 - \alpha r^2\right)^{\frac{1}{2}}\right]^2. \tag{1.35}$$

We see that $g_{00} = 4$ at $r = \alpha^{-1/2} \equiv R_h$ and that $g_{00} \geq 1$ for $0 \leq r \leq R_h$. In addition, if $r > R_h$, g_{00} is complex. Substituting the results of Eqs. (1.33), (1.35) and (1.16) into the metric line element of Eq. (1.7) gives

$$\begin{aligned} ds^2 = \left[2 - \left(1 - \frac{4\pi G\rho}{3c^2} r^2\right)^{\frac{1}{2}}\right]^2 c^2 dt^2 - \frac{1}{1 - \dfrac{4\pi G\rho}{3c^2} r^2} dr^2 \\ - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{1.36}$$

We note that this metric line element indicates that each point in space-time is surrounded by an event horizon located a spatial distance $r = R_h$ from it. At this horizon, g_{rr} becomes infinite and turns positive beyond, while $g_{00} = 0$ at the horizon and becomes complex beyond. This indicates that each point in space-time loses contact with the universe beyond its event horizon. Thus, this solution implies that while the universe is infinite, each point in space is connected to only a finite part of that universe. That is, this solution indicates that each point in space-time lives in its own complex black hole. Consequently, this infinite universe is "closed". Clearly then, this universe will be able to explain Olbers' paradox and so we tentatively accept it as a model for our Universe.

1.2.2 Nature of the Solutions

We have three solutions for a static isotropic homogeneous infinite universe; a "closed" universe with $\Lambda = \frac{4\pi G\rho}{c^2}$, a flat universe with $\Lambda = \frac{8\pi G\rho}{c^2}$, and an "open" universe with $\Lambda = \frac{12\pi G\rho}{c^2}$. We see from this that the cosmological constant Λ is needed in the theory of general relativity in order to obtain the solution for the three basic types of geometry. The constant achieves this by allowing the fitting of boundary conditions. Note, that as the density of the universe approaches zero, the cosmological constant Λ approaches zero for all three universes. However, as $\Lambda \to 0$, the solution for all three geometries coalesce into the flat-space solution of Minkowski space-time. In this sense the $\rho = 0$ solution for the infinite universe is degenerate. As we noted earlier, the Schwarzschild solution, which solves Einstein's equations assuming $\Lambda = 0$, represents the solution of an isolated mass in an otherwise empty universe. The method of our solution seems to indicate that the $\Lambda g_{\mu\nu}$ term should be included in the field equations for an exact solution to any matter distribution in an infinite matter filled universe.

The application of the three boundary conditions led to three different universes each of which possesses the flat space-time of special relativity locally near the origin. Each boundary condition can be seen to be the application of a different symmetry. The flat universe of Eq. (1.31) is associated with the planar symmetry of "the pull from one side cancels the pull from the other". The infinite open universe of Eq. (1.30) results from the application of spherical symmetry of infinite range while the infinite closed universe results from spherical symmetry with finite range. The different metrics of these universes will lead to different physics. If one of these universes matches the physics of our Universe then the corresponding symmetry will be the symmetry that holds in our Universe. The other two symmetries would then be only approximate.

The solutions we obtained for the infinite; "open", flat, and "closed" universes, respectively are:

$$ds^2 = \left(1+\omega r^2\right) c^2 dt^2 - \frac{1}{1+\omega r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (1.37a)$$

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (1.37b)$$

$$ds^2 = \left[2 - \left(1-\omega r^2\right)^{\frac{1}{2}}\right]^2 c^2 dt^2 - \frac{1}{1-\omega r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (1.37c)$$

with $\omega = \frac{4\pi G\rho}{3c^2}$. These metrics have the canonical general form of Eq. (1.7), indicating that these metrics are static and isotropic. Letting $k_- = -1, k_0 = 0$, and $k_+ = +1$, the spatial part of these metrics can be written

$$d\sigma^2 = -\frac{1}{1-k_-\omega r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (1.38a)$$

$$d\sigma^2 = -\frac{1}{1-k_0\omega r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \qquad (1.38b)$$

$$d\sigma^2 = -\frac{1}{1-k_+\omega r^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \qquad (1.38c)$$