CONTENTS

Preface 1
Introduction 5

1 MEASUREMENT CAPABILITY OF LANGMUIR PROBE 7
1.1 Ideal Planar Probe at Electron-Repelling Potential \( V_f < V < 0 \) in Plasmas with Isotropic Electron Distribution Function. 7
1.2 Ideal Cylindrical Probe at Electron-Repelling Potential \( V_f < V < 0 \). 11
1.3 Ideal Langmuir Probe at Particle-Attracting Potential \( eV < 0 \). 14
1.4 Entire Electron Branch of Probe I-V Characteristic in Isotropic Plasmas. Plasma Potential. 20
1.5 Plasma Referred Probes. 21
1.5a Symmetric Double Probe. 22
1.5b Slightly Asymmetric Double Probe. Triple Probe. 23
1.5c Strongly Asymmetric Double Probe. 24
1.5d Strongly Asymmetric Triple Probe. 25
1.6 Cylindrical Langmuir Probe in Plasmas with Electron Flows. 27
1.7 Planar Probe in Plasmas with Ion Flows. 35
1.8 Plasma Potential Measurements. Emissive Probes. 36
1.8a Inflection Point. 36
1.8b Differential Emissive Probe. 39
1.8c Secondary Emission Capacitive Probe. 40
1.8d Self-Emissive Probes. Disposable Probes. 42
1.9 Thermal Limits of Langmuir Probe. 43

References 47

2 REAL PROBE 47
2.1 Probe-Originated Distortions of Plasma Components Concentration in Near-Probe Vicinity. Probe Sheath. Method of Two Cylindrical Probes of Different Diameters. 47
2.1a Glow and high ionized plasmas, electron-collecting mode of probe at electron-repelling potential \( V_f \leq V \leq 0 \) at \( \lambda_D \approx \lambda_Te \). 49
2.1b Glow plasmas, electron-collecting mode of probe biased with electron-repelling potential \( V_f \leq V \leq 0 \) at \( \lambda_D \approx \lambda_Te \). 50
2.1c Glow plasmas and highly ionized plasmas, electron-attracting probe potential \( V \geq \varepsilon_p / e \) at \( \lambda_D \sqrt{1 - eV / \varepsilon_p} \approx \lambda_Te \). 50
2.1d Glow plasmas, electron-attracting probe potential \( V \geq \varepsilon_p / e \), \( \lambda_D \sqrt{1 - eV / \varepsilon_p} \approx \lambda_Te \). 52
2.1e Glow plasmas and highly ionized plasmas with \( \lambda_D \sqrt{1 - eV / \varepsilon_p} \approx \lambda_Te \), electron-attracting probe potential \( 0 \leq V < \varepsilon_p / e \). 52
2.1f Glow plasmas with \( \lambda_D \approx \lambda_Te \), electron-attracting potential of probe \( V \geq 0 \). 53
2.1g. Glow plasmas and highly ionized plasmas, ion branch of the probe I-V characteristic measured at the probe potential, $V < V_f - 2\varepsilon_p / e$, $C(V)\lambda_D \leq \lambda_{Te}$. 53

2.1h. Glow plasmas and highly ionized plasmas, ion branch of the probe I-V characteristic measured at the probe potential $V < V_f - 2\varepsilon_p / e$, $C(V)\lambda_D \geq \lambda_{Te}$. 54

2.1i. Plasmas with electron component in form of the fast electron beams, $V_f \leq V \leq 0$. 55

2.1j. Afterglow plasmas, electron-collecting mode of probe at electron-repelling potential $V_f \leq V \leq 0$. 55

2.1k. Afterglow plasmas, electron-collecting mode of probe at electron-attracting potential $V > 0$. 56

2.1l. Afterglow plasmas, ion branch of probe I-V characteristic $V < V_f - 2\varepsilon_p / e$. 57

2.2 Probe-Originated Distortion of Near-Probe Plasma Potential. “Impedance” of Near-Probe Plasma Induced by Probe Shadow. 58

2.3 Impedance of Afterglow Plasma to Probe Current. 66

2.4 Real Probe Operation in RF-Generated Plasmas. 70

2.5 Real Probe in Plasmas Generated in Magnetic Field. 72

2.6 Distortion of Probe I-V Characteristic Originated by Linear Ohmic Impedance of Long Cylindrical Probe. 74

2.7 Effects of Probe Surface Contamination. 77

References

3 PRACTICAL PROBE APPLICATIONS 83

3.1 Probe Diagnostics of Glow Discharge Plasma in Anode Column. 83

3.1a Experimental device and measurements procedure. 83

3.1b Comparative results obtained by two probes of different diameters at potential $V_f < V \leq 0$. 85

3.1c Reconstruction of undisturbed plasma parameters from probe data obtained at potential $V_f < V \leq 0$. 92

3.1d Reconstruction of undisturbed plasma parameters from probe data obtained at electron-attracting potential $V > 0$. 102

3.1e Reconstruction of undisturbed ion concentration from probe data obtained at ion branch, $V \ll V_f$. 109

3.1f Discussion. 113

3.2 Probe Measurements in Plasma of Crookes-Hittorf (Cathode) Dark Space in Coaxial Geometry of Gas Discharge. Problem of Electron Distribution Function. 113

3.2a Experimental device and technique of measurements. 114

3.2b Main experimental results and their interpretation. 117

3.3 Probe Diagnostics of Alkaline Plasma in Magnetic Fields (Q-Machine). 125
3.3a Experimental device and instrumentation. 125
3.3b Measurements of ion branches of “real probe” I-V characteristics. 127
3.3c Summary 132
3.4 Plasma of ITO (Indium-Tin-Oxide) Sputtering Magnetron in Ar-Atmosphere. 133
3.4a Description of experiments. 133
3.4b Experimental results. 134
3.5 Pulse Plasma with Electron Flows excited by Arc-Type Sources. 140
3.5a Experimental devices and probe arrangements. 141
3.5b Processing of probe I-V characteristics. 142
3.5c Summary. 152
3.6 Probe Measurements in High Density RF-Generated Plasma. 153
3.6a Ferrite enhanced inductive plasma source and probe arrangement. 154
3.6b I-V characteristics of self-emissive probes in steady dense RF plasma. 154
3.6c Discussion of results. 154
3.7 Probe Measurements in Flowing Afterglow Plasmas. 162
3.7a Experimental devices and probe arrangements. 164
3.7b Probe I-V characteristic at electron-repelling probe potential, \( V_f < V < 0 \), in afterglow plasma. 166
3.7c Probe I-V characteristic at electron-attracting probe potential, \( V > 0 \), in afterglow plasma. 175
3.7d Ion branch of probe I-V characteristic, \( V < V_f \), in afterglow plasma. 182
3.7e Intermediate part of probe I-V characteristic in afterglow plasma. 184
3.7f Probe diagnostics of afterglow plasmas, summary. 191
References

4 PROBE CONFIGURATIONS AND CIRCUITS FOR PROBE ACTIVATION, CLEANING, AND MEASUREMENTS 197
4.1 General Aspects of Probe Design. 197
4.2 General Problems of Probe Activation and Circuit for Rod-Probe. 201
4.3 Circuit for Plasma-Referred Probe Operating in High Density Plasmas. 202
References

5 THERMOCOUPLE MEASUREMENTS OF NEUTRAL GAS TEMPERATURE IN SLIGHTLY IONIZED GASES 207
5.1 Reasons for Measurements of Neutral Gas Temperature in Slightly Ionized Gases. 207
5.1a Microthermocouple for gas temperature measurements in slightly ionized gases. 208
5.2 Experiments on Microthermocouple Calibration in Glow Discharge. 209
References.

CONCLUSION 213
ACKNOWLEDGMENTS 215
References
Appendices

Appendix A  Parameter $\langle v \rangle / v_{de}$ in Afterglow Plasmas 218

Appendix B  Derivation of Electron Temperature from Probe I-V Characteristic at Electron-Repelling Potential. 219

B1  Maxwellian Distribution Function. 219

B2  Maxwellian Distribution Function Shifted and Diffused. 219

B3  Druyvestein’ Distribution Function Shifted and Diffused. 224

B4  Maxwellian Distribution Function Shifted. 228

References

Appendix C  Table of Specific Impedance of Near-Probe Plasma. 235

Appendix CC  Faraday Cup 239

Appendix D  Using the Programs Attached. 243
This work is intended to describe the operation of the Langmuir Probe in various plasmas at different conditions. Physical and mathematical descriptions presented in this book assume a university-level background for readers. The author presumes that this book will be used as a text book in a plasma–diagnostics part of a plasma–physics course by undergraduate and graduate level students, and as a practical reference manual by physicists and engineers working with plasma.

For definition purposes, let us attach the name “Langmuir Probe” to a small size electrode submerged in plasma and connected through an external (with respect to plasma) electrical circuit with a conductor of a large surface area in contact with this plasma (very often this large conductor is the metallic wall of a vacuum chamber) if the I-V characteristic (current-voltage characteristic) obtained by the voltage scanning in the mentioned external electrical circuit is processed for plasma parameters derivation.

In light of this definition, one can assert that Langmuir Probe Plasma Diagnostics originated in 1926 with the publication of the famous paper of Irving Langmuir and H. M. Mott-Smith. In that paper, the authors presented a derivation of the expression connecting the electron distribution function in plasma with the I-V characteristic of the probe submerged in this plasma. In 1930, M. J. Druyvestein published a simplified derivation of expressions describing plasma-probe interaction in assumption of a complete spherical isotropy of the electron distribution function. In addition, Druyvestein demonstrated that the second derivative of the probe I-V characteristic with respect to the probe potential is proportional to the electron distribution function in plasma for various convex shapes of probes.

There is a large arsenal of plasma diagnostic methods discovered and developed since the original work of the Irving Langmuir publication. However the Langmuir Probe Diagnostics are still a powerful tool in laboratory practice due to their simplicity, inexpensiveness, easy adaptiveness to specific needs, the ability to perform local measurements of the electron distribution function and plasma potential at a good time resolution (~10^-8 seconds), and also (in a light of all the above reasons) the technical possibility of obtaining the requested results quickly in the range of a plasma density 10^3 – 10^14 cm^-3 with an average electron energy from 0.025 eV (room temperature) to hundreds of electron-Volts. Thus in spite of the contact nature of the probe measurements, a low expected precision of data measured (owing to a slightly developed theory of a “real” probe operation), and a limited upper range of measured plasma densities and temperatures (due to a limited thermal stability of the probe material), Langmuir Probe Diagnostics continues to be one of the most popular instruments for physicists and engineers working with plasmas. It is useful to note that about 500 technical papers describing Langmuir probe applications were published from 2000 to 2007 alone, in leading scientific journals.

A diversity of theoretical and experimental works based on the results obtained by the Langmuir Probe Diagnostics are available presently. However there are very few publications dedicated to processes undertaken in the near-probe vicinity. Among these publications, it seems important to mention the work of E. Johnson and L. Malter, who developed the Symmetric Double Probe in 1949; and the work of N. Hershkowitz with its careful and
detailed review of the problems and important developments in the Langmuir Probe theory and applications. In this work, Hershkowitz stressed the importance of taking into account (or at least of remembering) various types of distortions induced in plasmas interfering with the “real” probe. Numerous publications of F. F. Chen 6 concerning various aspects of the Langmuir probe theory and applications have also become a significant part of contemporary Langmuir probe diagnostics. The author of the present book apologizes if he does not mention in this short essay all the scientists who participated in building the probe diagnostics methods to their contemporary form. Their works are reflected in corresponding parts of this book and cited with the author’s gratitude.

The author utilizes more than 30 years of experience in theory and practical applications of Langmuir Probes to reveal and describe various differences between an ideal Langmuir Probe operating in an imaginary undisturbed-by-the-probe plasmas and a real probe consuming actual electric current (and electrical charges!) sunk from surrounding probe plasma into the probe electrical circuit 7-9. Briefly, some of the differences and results are noted here:

1. It was found that the less geometrical size of the probe consuming the less electric current, the more (and closer to undisturbed level) the value of plasma component density derived by conventional Langmuir expressions from the probe data.

2. Measurements in plasmas of the anode column of a glow discharge 7 and in plasmas with a high percentage of ionized components 10 revealed that the shape of the probe I-V characteristic at the electron–repelling potential of the probe depends in the certain cases on the probe size (the probe diameter for cylindrical probes). This dependence occurs due to a predominant outflow of particles with the certain electric charge from the probe vicinity into the probe electrical circuit and, consequently, leads to a relative increase of the redundant electric charge of the opposite sign, inducing a corresponding potential bias for the near-probe plasma.

3. In light of the near-probe plasma distortions induced by the probe current, experiments testing the correctness of the Langmuir probe theory were performed 8. These experiments enabled the author to study plasma of Crooks-Hittorf near-cathode space and corroborate experimentally the basic points of theoretical results obtained by Langmuir and Druyvestein.

4. There were successful attempts 9,11 to derive the electron and the ion concentration in plasmas from branches of the same probe I-V characteristic at the electron–repelling, electron–attracting, and the ion–attracting probe potentials.

5. I-V characteristics of the Langmuir Probe in plasmas with the electron beams (runaway electrons) flowing about the probe with average velocities commensurable with or exceeding the average chaotic thermal electron velocity (determined in a reference system moving with the average velocity of these beams) were revealed in the experiments and are described theoretically 10,12.

6. It was found experimentally that the I-V characteristics of the Langmuir Probe operating in afterglow plasmas are significantly different in their shapes from the I-V characteristics of the probe operating in the anode column plasmas of a glow discharge or in highly ionized plasmas. That difference occurs because of a different character of the charged particles delivery from the surrounding probe plasma to the probe vicinity and also with the specific features of cold plasmas with temperatures of their components close to room temperature 9.
7. The author had intensive experience in working with high-density RF-excited plasmas for numerous industry applications including semiconductor and display production technologies. The probes interacting with high-density steady plasmas demonstrated significant differences in their I-V characteristics at the electron-attractive potential of probe operation in comparison with “cold” probes.

8. The probes operating in plasmas developed by magnetrons spattering oxides showed a significant decrease of the electron-collected currents measured at the electron-repelling probe potential due to a perceptible secondary electron emission of oxides coating the probe surface extremely quickly after the probe cleaning procedure.

In light of the preceding developments, the author found it reasonable to collect all the Langmuir Probe Diagnostics results, obtained by him in a long journey of experimental and theoretical investigations, under one common roof, together with a description of the probe configurations and electronic circuitry developed for probe I-V characteristics measurements.

Another reason for consolidating all this research in a single volume is to describe in detail the derivations of expressions deduced for probe I-V characteristics processing in the specific cases, and to provide the reader with all the available information concerning practical applications of these expressions.
INTRODUCTION

Chapters 1 through 5 comprise in part the results of the author’s experience in Langmuir Probe Diagnostics as well as a detailed review of the most reliable probe diagnostics in theory and practice, including data processing, probe fabrication, and electric circuitry design. Chapter 1 contains information on plasma parameters capable of being measured by the probe, and also a technique for processing the I-V characteristic of the “ideal” probe to obtain the plasma parameters required. In addition, Chapter 1 supplies the requisite physics information for an understanding of the principles of probe operation. For the highly educated reader, the material in this chapter will be useful in most part for review, although some derivations and their consequences will be new for most readers.

Chapter 2 discusses how the real probe disturbs plasma components (electrons and ions) concentrations and, consequently, disturbs plasma potential in the near-probe vicinity. The methods of plasma parameters reconstruction are considered at distinctive modes of the probe operating in plasmas with various characteristics. Some of the details of the plasma parameters reconstruction procedure will be new for all readers.

Chapter 3 describes seven distinctive and most typical experimental cases of practical probe applications in different plasmas. These detailed descriptions are accompanied with real probe data obtained in experiments, examples of this data processing, and methods of their reliability examination. Two sections of this Chapter discuss probe measurements in plasma of Indium-Tin-Oxide sputtering magnetron and in high density RF-generated plasma. These contain significant new material.

Chapter 4 covers probe configurations and some detail of probe design and fabrication, as well as electric circuits for probe activation, cleaning, and measurements. Chapter 5 describes a technique of thermocouple measurements in slightly ionized plasmas and offers a method of verification of gas-temperature measurements reliability.

Four appendices support the material covered in the text. Among them, Appendix B describes a technique of probe I-V characteristic processing for deducing electron temperature and electron velocity in plasmas. Sections B2 and B3 of this appendix contain new information that enables the reader to evaluate the electron distribution function in certain kinds of plasmas as the Maxwellian or Druyvestein distribution function shifted by the action of the electric strength in plasma and diffused on plasma components to almost isotropic shape. The same descriptions enable one to simplify a procedure of the probe I-V characteristics processing and to facilitate practical applications of the experimental electron distribution function in various plasma-connected calculations. Appendix C presents a table of the specific plasma resistance induced in the probe vicinity as a consequence of the electron deficiency caused by the probe operation. Appendix D, contained of program file available at the publisher’s website, provides nine programs for processing the electron branches of the probe I-V characteristics obtained in the eight most frequently occurring cases of the probe application, along with a Library of nine experimental probe I-V characteristics measured in these cases; and the file “ReadMeD1.doc”, describing briefly the operation of the programs. All the programs have been written by VBA6 and can be used with any PC configuration provided with EXCEL.
References

1 MEASUREMENT CAPABILITY OF LANGMUIR PROBE

1.1 Ideal Planar Probe at Electron-Repelling Potential $V_f \leq V \leq 0$ in Plasmas with Isotropic Electron Distribution Function.

The Langmuir Probe is the ideal measuring instrument if the electric current in its circuit is negligibly small. This requirement assumes that the outflow of charged particles from the probe vicinity to the probe electric circuit does not affect plasma component density anywhere around the ideal probe. Initially, for the purpose of simplification, we will investigate the ideal planar probe with an infinitesimally large surface area faced to plasma. This means we will neglect the probe edge effects. We will consider the I-V characteristic of the probe in the electron–repelling potential range, $V_f \leq V \leq 0$, where the electron current on the probe is still more than the ion current due to a significant difference in a mobility of the electrons and ions in plasmas. Here $V_f$ is the floating potential of the probe, that is, the potential at which the sum of the electron- and the ion currents on the probe surface from plasma is equal to zero. This section will describe the probe operation in plasmas with sufficiently isotropic electron distribution functions having a complete spherical symmetry in velocity space.

![Image](image.png)

**Figure 1.** Illustration to the derivation of the I-V characteristic of the planar Langmuir probe.

To start, we will follow derivations obtained by Langmuir and Druyvestein under the conditions described above. For this purpose, let us choose the point $O$ at the certain remote distance from a surface of the mentioned planar probe (see Fig.1), where the electric field of the probe can be neglected however collisions with plasma components of plasma electrons moving along any direction from this point to the probe surface would be negligibly small.
This requirement can be realized approximately for an *ideal* probe if the Debye length $\lambda_D$ in the plasma is sufficiently less than the free electron path $\lambda_{Te}$ defined by a value of a total cross section $\sigma_{Te}$ of the electrons scattering on plasma components (including a neutral gas), $\lambda_D \ll \lambda_{Te}$. In a vicinity of the point $O$, let us chose the infinitesimally small element of the surface area $dS$ parallel to the probe surface. The elementary current of plasma electrons passing throughout this surface area in the direction perpendicular to the probe surface can be written down in the form

$$di = e \Delta S dn(v, \vartheta) v \cos \vartheta$$

where $v$ is the scalar of the electron thermal velocity vector $\mathbf{v}$,

$$dn(v, \vartheta) = n f(v) \frac{2\pi \sin \vartheta}{4\pi} dv d\vartheta,$$

$2\pi \sin \vartheta d\vartheta$ is the element of the solid angle while $2\pi \sin \vartheta d\vartheta/4\pi$ is its relative value, $\vartheta$ is the angle between a normal to the probe surface and the radius-vector of the electron thermal velocity $\mathbf{v}$ forming a spherical layer of thickness $dv$ in velocity space, and $f(v)$ is the electron distribution function normalized to the unity. Because the location of the point $O$ in the coordinate system of the probe plane is chosen arbitrarily, the expression for the total probe current can be written down in the form

$$i(V) = enS_z \frac{1}{4\pi} \int_{\sqrt{2eV/m}}^{\infty} f(v) dv \int_{0}^{\zeta} v \cos \vartheta \ 2\pi \sin \vartheta d\vartheta.$$

(1.1)

where $V$ is the probe potential with respect to the plasma potential $V = 0$, $\sqrt{2eV/m}$ is the lowest value of the electron velocity at which the electron still could reach the surface of the probe having the repelling potential $V$, $\zeta$ is the upper limit of the angle $\vartheta$ at which the electron having initial velocity $\mathbf{v}$ can still reach the probe surface with the zero-value of its velocity at this surface (that means the value of $\zeta$ is defined by the condition

$$v \cos \zeta = \sqrt{2eV/m}$$

(1.2)

$n$ is the electron concentration, $S_z = \int dS$ is the probe surface area, and the electron distribution function $f(v)$ in Eq. (1.1) is normalized by the condition

$$\int_{0}^{\infty} f(v) dv = 1.$$

(1.3)

Deriving the value $\zeta$ from Eq. (1.2) and substituting it in Eq. (1.1), we can obtain the probe I-V characteristic (neglecting the ion current) in the range of the probe potential $-\infty < V \leq 0$ in the form

$$i(V) = \frac{enS_z}{4} \int_{\sqrt{2eV/m}}^{\infty} f(v) \left(1 - \frac{2eV}{mv^2}\right) v dv.$$

(1.4)

Differentiating the Equation (1.4) with respect to the potential $V$, one can find the first derivative of the probe I-V characteristic:

$$i'(V) = -\frac{enS_z}{4} \frac{2e}{m} \int_{\sqrt{2eV/m}}^{\infty} \frac{f(v)}{v} dv,$$

(1.5)
and, after the second differentiating, obtain the expression for the second one:

\[ i''(V) = \frac{e^2 n S_z}{4m} \frac{1}{V} f\left(\sqrt{2eV/m}\right). \]  

(1.6)

The derivation of the Equations (1.1), (1.4), (1.5), and (1.6) presented here has been followed the Druyvestein’s original work. In works of Druyvestein, it was shown that the Equations (1.4), (1.5), and (1.6) hold true for the cylindrical, spherical, and any convex geometry of the probe surface as well. Note that in the work of the Langmuir, the Descartes’ coordinates in velocity space were used for the purpose of the Equation (1.6) derivation. It is important to emphasize that the probe I-V characteristic described by Eq. (1.4) at the electron–repelling probe potential does not depend on a character of the electric field distribution between the probe and near–probe plasma, in contrast to the cases of the electron–attracting or ion–attracting probe potentials.

It follows directly from Eq. (1.4) that the electron current on the probe at the potential \(V = 0\) can be expressed in the form:

\[ i(0) = \frac{en\langle v \rangle}{4} S_z, \]  

(1.7)

where

\[ \langle v \rangle = \int_{0}^{\infty} f(v) v dv \]  

(1.8)

is the thermal electron velocity averaged over the electron distribution function \(f(v)\) by definition; see Eq. (1.3).

Substituting the Maxwellian distribution function:

\[ f_M(v) = \frac{4}{\sqrt{\pi}} \frac{v^2}{v_p^3} \exp\left(-v^2/v_p^2\right) \]  

(1.9)

in Eq. (1.4), where \(v_p = \langle v \rangle \sqrt{\pi}/2\) is most probable velocity, we find the expression:

\[ i_M(V) = \frac{en\langle v \rangle}{4} S_z \exp\left(-eV/\varepsilon_p\right), \]  

(1.10)

from which the very useful practical Equation (see Eq. (1.4)) follows:

\[ \ln\left(i_M(V)/i_M(0)\right) = -eV/\varepsilon_p \]  

(1.11)

allowing one to derive the electron energy \(\varepsilon_p = kT\) (for Maxwellian distribution function only!) corresponding to the most probable velocity \(v_p\) by a slope of the probe I-V characteristic in a semi–logarithmic scale.

The point \(V = 0\) pertaining to the I-V characteristic of the “ideal” probe corresponds to the point where the probe electron–repelling mode is changed to the electron–attracting. This point coincides with the point where the probe I-V characteristic second derivative \(i''(V)\) crossing the abscissa axis changes its sign (see Section 1.4). The same assertion holds true for the I-V characteristics of the probe operating in any plasmas where the actual electron flow (non-induced by the probe operation) provides the plasma density maintenance in the probe vicinity. Thus the sink of the electrons into the probe–operating circuit is compen-
sated for by the mentioned electron flow (that is simply the electron drift in the electric fields of plasma) rather than by the electron concentration gradient caused by the probe operation as it occurs in afterglow plasmas.

In afterglow plasmas, the point $V = 0$ can be found as the point of a sharp change of the probe $I$-$V$ characteristic second derivative (see Section 3.7).

Thus applying scanning voltage $V(t)$ to the probe and measuring the probe current $i(V)$ in the probe circuit, one can derive consequently the **electron distribution function** $f(v)$ by Eq. (1.6), the **average electron velocity** by Eq. (1.8), and the **electron concentration** $n$ by Eq. (1.7).

From this it follows that one can obtain the complete portrait of the electron population in plasma. For plasmas with the Maxwellian distribution function, the value of the **electron temperature** $T_e$ can be derived from the slope of the probe $I$-$V$ characteristic in the semi-logarithmic scale; see Eq. (1.11).

The $I$-$V$ characteristic of the Langmuir probe (in isotropic plasmas with the Maxwellian electron distribution function) described by Eq. (1.10) is shown at $eV / kT_e \leq 0$ in Fig. 2.

**Figure 2.** The $I$-$V$ characteristic, $i(V)$, its first, $i'(V)$, and second, $i''(V)$, derivatives, of an ideal probe in plasma with the isotropic Maxwellian electron distribution function.

The $I$-$V$ characteristic of the Langmuir probe (in isotropic plasmas with the Maxwellian electron distribution function) described by Eq. (1.10) is shown at $eV / kT_e \leq 0$ in Fig. 2.
1.2 Ideal Cylindrical Probe at Electron-Repelling Potential $V_f < V < 0$.

As it was mentioned in the previous section, a geometrical shape of the ideal probe should not affect the probe I-V characteristic. However, the cylindrical geometry of the probe is most preferable for practical applications for four reasons:

First, it seems obvious that an uncollisional electron path to the planar probe surface is impossible for any reasonable distance $h$ from the chosen point $O_1$ to the probe surface, because of a path of the certain part of the electrons from the point $O_1$ to the probe surface will always longer than the any electron free path $\lambda_{Te}$ at large incident angles:

$$\zeta > ar\cos(h / \lambda_{Te}),$$

see Eq. (1.1). However, the situation looks remarkably encouraging for the cylindrical probe because this probe can consume the electrons having high incident angles $\zeta \approx \pi / 2$ only within the narrow azimuth angles $\psi$ in vicinities of the directions of the cylindrical probe axis (see Fig. 3). Additionally, the values of these narrow angles are decreased drastically with decrease of the probe diameter.

Second, the practical design of the probe with accompanying interconnecting wires is simplified for the cylindrical probe significantly. This fact is especially important when taking into account a necessity of the Ohmic warming up for probe-surface cleaning or for obtaining the emissive probe effects.

Third, a “plasma–disturbing” size of the probe (in the cylindrical case it is a diameter of the probe) can be easily decreased to some reasonably small value.

And fourth, the thin cylindrical probe can be fabricated of wire with a diameter that allows negligibly slight depletion of the electron flux tubes in plasmas in the presence of a magnetic field. Therefore we assume a restriction to study the cylindrical probe operation mostly, otherwise specified.

Let us utilize Eq. (1.4) specifically for the cylindrical probe geometry that assumes an infinitesimally long (to neglect edge effects) cylindrical probe. For this purpose, we choose the coordinate system shown in Fig. 3.

The point $O_1$ is distanced from the probe axis $x$ at the certain radius $R$ where the electric field of the probe (its potential is negative) is negligibly small while this distance is significantly less than the electron free path. The plane $p$ is orthogonal to the radius–vector $r$ and includes the point $O_1$. In the coordinate system derived from this method, one can obtain the equation of energy conservation for the electrons moving from the point $O_1$ to the probe surface (all parameters supplied with indices “z” correspond to the probe surface) in the form

$$v_r^2 + v_z^2 = v_{rz}^2 + v_{rz}^2 + 2eV / m ,$$

and the equation of the angular momentum conservation in the form

$$R v_r = r_z v_{rz} .$$

Here $v_r$ is the $r$–component of the electron thermal velocity vector $v$ taken along a negative direction of the radius–vector $r$ in the point $O_1$:

$$v_r = v \cos \vartheta ,$$

and $v_r$ is the “tangent” component of $v$ that is component taken along the direction of the vector $x \times r$ in the point $O_1$:

$$v_r = v \sin \vartheta \sin \psi ,$$

Here $v$ is the electron thermal velocity in the point $O_1$.
where \( v \) is the scalar of the electron thermal velocity \( \mathbf{v} \) vector, \( \vartheta \) is the angle between the vector \( \mathbf{v} \) and the negative direction of \( \mathbf{r} \), \( \psi \) is the angle between a projection of \( \mathbf{v} \) on the plane \( p \) and a projection \( x_1 \) of the probe axis \( x \) on the same plane (azimuth angle), and \( v_{rz} \).

The \( x \)-component of the thermal velocity is conserved for the electron moving in the electric field of the probe, and it does not contribute any value into the probe current.

In terms described above, the electron probe current can be described by the Equation

\[
i(V) = 2\pi Rl_en \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} d\psi \int_{0}^{\infty} f(v) dv \int_{0}^{\xi} \nu \cos \vartheta \sin \vartheta \ d\vartheta .
\] (1.16)

Figure 3. Illustration to the derivation of the I-V characteristic of the cylindrical probe.

and \( v_{rz} \) are the values of the \( v_r \) and \( v_r \) at the radius \( r = r_z \), where \( r_z \) is the probe radius. The \( x \)-component of the thermal velocity is conserved for the electron moving in the electric field of the probe, and it does not contribute any value into the probe current.

In terms described above, the electron probe current can be described by the Equation

\[
i(V) = 2\pi Rl_en \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} d\psi \int_{0}^{\infty} f(v) dv \int_{0}^{\xi} \nu \cos \vartheta \sin \vartheta \ d\vartheta .
\] (1.16)

Here \( \nu \cos \vartheta \) is the electron velocity component forming the probe current; the electron distribution function \( f(v) \) is normalized to unity (see eq. (1.3)); the representation \( \nu \)

\[
\frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} F(\psi)d\psi = \frac{1}{4\pi} \int_{0}^{2\pi} F(\psi)d\psi
\]

is applied due to apparent angular symmetry of the integrand \( F(\psi) \), see Eq. (1.16), where factor \( 1/4\pi \) is responsible for averaging over the solid angle (see derivation of Eq. (1.1)); \( \xi \) is the limit of the angle \( \vartheta \) in the case where the radial component of the thermal velocity vector \( \mathbf{v} \) approaches to its zero value at the probe surface \( v_{rz} \rightarrow 0 \); and \( 2\pi Rl_z \) is the electron-
emitting surface. To determine $\zeta$, one can substitute Eq. (1.13) in Eq. (1.12), taking into account Eqs. (1.14) and (1.15), resulting in the relation

$$
\zeta = \arccos \left( \frac{a^2 \sin^2 \psi + 2eV / mv^2}{1 + a^2 \sin^2 \psi} \right)^{1/2} ,
$$

(1.17)

where

$$
a^2 = \left( R^2 - r_z^2 \right)/r_z^2 .
$$

(1.18)

Substituting Eq. (1.17) in Eq. (1.16) and performing integration with respect to $\vartheta$, we obtain the Equation

$$
\frac{i(V)}{2\pi R l_e n} = \frac{1}{2\pi} \int_{0}^{\pi/2} d\vartheta \int_{\sqrt{2eV/m}}^{\infty} f(v) v dV \left( 1 - \frac{a^2 \sin^2 \psi + 2eV / mv^2}{1 + a^2 \sin^2 \psi} \right) ,
$$

(1.19)

reducible to the form

$$
\frac{i(V)}{2\pi R l_e n} = \frac{1}{2\pi} \int_{0}^{\pi/2} \left( 1 - a^2 \sin^2 \psi \right)^{-1} d\vartheta \int_{\sqrt{2eV/m}}^{\infty} f(v) \left( 1 - \frac{2eV}{mv^2} \right) v dv .
$$

(1.20)

Taking into account that

$$
\int_{0}^{\pi/2} \left( 1 - a^2 \sin^2 \psi \right)^{-1} d\vartheta = \left( 1 + a^2 \right)^{1/2} \arctg \left( \sqrt{1 + a^2} \tan \vartheta \right) \right|_{0}^{\pi/2} = r_z \frac{\pi}{R} ,
$$

(1.21)

because of

$$
1 + a^2 = \frac{R^2}{r_z^2} ,
$$

(see Eq. (1.18)), one can rewrite Eq. (1.20) in the form

$$
i(V) = \frac{enS_z}{4} \int_{\sqrt{2eV/m}}^{\infty} f(v) \left( 1 - \frac{2eV}{mv^2} \right) v dv ,
$$

(1.22)

which is completely identical with the form described by Eq. (1.4). Here $S_z = 2\pi r_z l_z$ is the operating surface area of the cylindrical probe. It is understood that the Equation (1.22), derived in negligence of the ion current, holds true actually in the voltage range $V_f \lesssim V \lesssim 0$.

Thus we have the exact proof that the I-V characteristic of the “ideal” probe has, in the voltage range $-\infty < V < 0$, the identical expression for both the planar probe of the infinite surface and for the cylindrical probe of any arbitrary radius and infinite length. Therefore all the equations derived for the planar probe (Eqs. (1.5) – (1.11)) are applicable for the cylindrical probe as well. Besides that one can assert in view of Eq. (1.22) derivation that the general shape of the I-V characteristic obtained by the cylindrical probes in the same plasma should not depend on the probe radius at all.

The angular momentum conservation reflected by Eq. (1.13) assumes that electrons moving from the arbitrarily chosen radius $R$ to the probe surface (with the radius $r_z$) do not have intermediate collisions accompanied with the certain loss in the electron momentum. In addition, the equation of the energy conservation (Eq. (1.12)) was written under the assumption
that the electric field of the probe at the chosen distance $R$ from the probe axis was negligibly small. Therefore the logical result following from both these limitations can be written as the only one actual requirement:

$$\lambda_D \leq \lambda_{Te}$$

restricting the Equation (1.22) application.

### 1.3 Ideal Langmuir Probe at Particle-Attracting Probe Potential $eV < 0$.

Reliable derivation of the electron concentration $n$ and the electron distribution function $f(v)$ by processing a part of the probe I-V characteristic at the electron-repelling potential ($V_f < V < 0$) is a simple procedure only at the relatively low level of the electromagnetic interferences and instabilities of plasma accompanied with a relatively high level of a signal from the probe. However in usual cases, the experimentalist could only dream about those comfortable conditions. If the processing of the probe I-V characteristic parts at the electron-attracting $V > 0$ and/or the ion-attracting $V < V_f$ probe potentials could give us the reliable and precise values of the electron (or ion) concentration, it would broaden the probe diagnostics seriously to the ranges of increased noise-signal ratios due to the obvious possibility of obtaining a higher electrical response from the probe and to achieve a simplified method to average electronically the probe signal over a specified time.

A problem for the theoretical consideration of the particle-attracting probe in plasmas is extremely difficult because a current collected by the probe at this mode of its operation depends in general case on a character of the potential distribution over the probe vicinity (in contrast to the particle-repelling probe $^8$, see Sections 1.1 and 1.2). An expression allowing one to perform the rough estimation of the ion saturation current on a spherical ion-attracting probe was found by Bohm in a frame of plasma ambipolar diffusion assuming a negligible small ion temperature $^9$

$$i_{is} \approx 0.6 n_{\infty} S_{z} \sqrt{\langle \varepsilon_e \rangle / M_i}, \quad (1.23)$$

where $i_{is}$ is an ion saturation current, $\langle \varepsilon_e \rangle$ is the average electron energy, and $M_i$ is ion mass. Bohm’s theory was developed by Allen et al $^10$. The ion temperature was taken into account by Bernstein and Rabinowitz $^11$, Chen $^12$, and Lam $^13$. The results of those works were obtained by solving a self-consistent Poisson equation for a near-probe layer of plasma and some of them include a capture of charged particles into a vicinity of the cylindrical and the spherical probes $^4,14$. However it is obvious that, for example, in the anode column plasma of a glow discharge within the strong own electric fields and the sufficiently particle-scattering discharge nature, the capture of the charged particles in the orbital motion around the probe installed transverse to the discharge current (and consequently transverse to the electric field in plasma) seems very problematic.

Because a detailed analysis of the potential distribution in the near-probe plasma-layer of the real cylindrical probe is very difficult, we will neglect in our consideration the actual character of this distribution in the first order of approximation. We will assume that attracted particles have an isotropic and spherically symmetric distribution function in velocity space, and this symmetry is conserved near the probe surface in a reference system moving with velocity $\nu_z = \sqrt{2eV/\mu}$ acquired in the probe electric field $E(r)$. Here $\mu$ is the mass of the
attracted particle, and $V$ is the probe potential with respect to plasma potential equal to zero. Note that this physical pattern can take place only at the condition $\lambda_d < \lambda_T$. We will restrict our analysis to the ideal planar probe with the infinite surface. However in view of the fact that all the experimental data presented in this book were obtained by a thin cylindrical probe having a radius $r \ll \lambda_d$, we will take into account both semi-spheres of the angular particle distribution in velocity space as is shown in Fig. 4.

![Figure 4](image)

**Figure 4.** Illustration to the derivation of the probe I-V characteristic at the particle-attracting potential.

Under these conditions, the current on the probe at the attracting potential $V$ can be presented as three components. The upper (above the probe surface, see Fig. 4) semi-sphere of velocity space is responsible for one of them. The velocity range of particles for this component is $0 \leq v \leq \sqrt{-2eV/\mu}$; here the signs of the potential and the charge are taken into account. The orthogonal to the probe surface component of the particle velocity is $v_\perp = \sqrt{-2eV/\mu - v \cos \theta}$, where $v$ is the scalar of the thermal velocity $\mathbf{v}$, and $\theta$ is the angle between the vector $\mathbf{v}$ and the normal to the probe surface. Thus this part of the particle distribution forms the current $i_{+1}$ on the probe described by the Equation (cf. derivation of Eq. (1.1)):

$$
\frac{i_{+1}(V)}{enS} = \frac{1}{2} \int_0^{\pi/2} f(v) \sin \theta d\theta \int_0^{\pi/2} (\sqrt{-2eV/\mu - v \cos \theta}) \sin \theta d\theta, \quad (1.24)
$$

whence
\[
\frac{i_{12}(V)}{enS_z} = \frac{1}{2} \int_{0}^{\frac{-2eV/\mu}{2}} f(v) \left( \frac{-2eV}{\mu} - \frac{v}{2} \right) dv.
\]

Here

\[
\frac{1}{2} \sin \theta d\theta = \frac{2\pi \sin \theta d\theta}{4\pi}
\]
is the element of the solid angle \(2\pi \sin \theta d\theta\) normalized over the total solid angle \(4\pi\).

The upper semi-sphere of velocity space is responsible also for the second current component \(i_{12}\). However the velocity range in this case is \(\sqrt{-2eV/\mu} \leq v \leq \infty\), so that:

\[
\frac{i_{12}(V)}{enS_z} = \frac{1}{2} \int_{\frac{-2eV/\mu}{2}}^{\infty} f(v) dv \int_{\arccos(\frac{-2eV/\mu}{v})}^{\frac{\pi}{2}} \left( \sqrt{-2eV/\mu} - v \cos \theta \right) \sin \theta d\theta,
\]

whence

\[
\frac{i_{12}(V)}{enS_z} = -\frac{1}{4} \int_{\frac{-2eV/\mu}{2}}^{\infty} f(v) \frac{2eV/\mu}{v} dv.
\]

The lower semi-sphere of velocity space is responsible for the third component of the current \(i_{33}\). The component of \(v\) orthogonal to the probe surface is \(v_\perp = v\cos \alpha + \sqrt{-2eV/\mu}\), where angle \(\alpha = \pi - \theta\) is the complimentary one to the incident angle \(\theta\). Thus we obtain the Equation

\[
\frac{i_{33}(V)}{enS_z} = \frac{1}{2} \int_{0}^{\infty} f(v) dv \int_{0}^{\frac{\pi}{2}} \left( \sqrt{-2eV/\mu} + v \cos \alpha \right) \sin \alpha d\alpha,
\]

convertible to the sum:

\[
\frac{i_{33}(V)}{enS_z} = \frac{1}{2} \sqrt{-2eV/\mu} + \frac{\langle v \rangle}{4},
\]

where the definition of the electron velocity average value

\[
\langle v \rangle = \int_{0}^{\infty} v f(v) dv
\]
determined by the distribution function normalization procedure:

\[
\int_{0}^{\infty} f(v) dv = 1
\]
is taken into account.

Summarizing all the three derived above probe current components expressed by Eqs. (1.25), (1.27), and (1.29) yields the Equation:

\[
\frac{i_{12}(V)}{enS_z} = \sqrt{-2eV/\mu} + \frac{1}{4} \int_{\frac{-2eV/\mu}{2}}^{\infty} f(v) v \left( 1 - \sqrt{\frac{-2eV/\mu}{v}} \right) dv,
\]

comprising two clear cut terms in its right-hand part. Taking into account that the potential variations along the normal to the probe surface were ignored, and understanding that both terms in the right-hand part of Eq. (1.30) have obvious asymptotes at \(V \to 0\) and at \(V \to \infty\), one can expect that two invariable dimensionless coefficients \(A\) of the first- and \(B\) of the
second term in Eq. (1.30) could give us a good description of the probe I-V characteristic at the attracting potential. Under all these assumptions, we get the Equation

\[
\frac{i_+(V)}{enS_z} = A\sqrt{-2eV/\mu} + \frac{B}{4} \int_{-\infty}^{\infty} f(v) \left( 1 - \frac{\sqrt{2eV/\mu}}{v} \right)^2 dv. \tag{1.31}
\]

To find the values of the coefficients \(A\) and \(B\) entered in Eq. (1.31), we can use the continuity of \(i(V)\) and \(i'(V)\) in the point \(V = 0\) conformed by numerous experimental facts. From the equation \(\lim_{V \to 0} i_+(V) = \lim_{V \to 0} i(V)\) (see Eqs. (1.31) and (1.4)), the value \(B = 1\) follows directly. Differentiating the Equation (1.31) with respect to the variable \(V\), one can obtain the Equation:

\[
\frac{i'_+(V)}{enS_z} = -\frac{A}{2} \frac{2e}{\mu\sqrt{-2eV/\mu}} + \frac{B}{2} \frac{2e}{\mu\sqrt{-2eV/\mu}} \int_{-\infty}^{\infty} f(v) \left( 1 - \frac{\sqrt{2eV/\mu}}{v} \right) dv. \tag{1.32}
\]

Comparing the Equation (1.5) with Eq. (1.32) at \(B = 1\) and at \(V \to 0\), one can derive the value \(A = 1/2\). Substituting further \(A = 1/2\) and \(B = 1\) in the Equation (1.31), one can find the final form for the probe I-V characteristic at the particle-attracting potential:

\[
\frac{i_+(V)}{enS_z} = \frac{1}{2} \sqrt{-2eV/\mu} + \frac{1}{4} \int_{-\infty}^{\infty} f(v) \left( 1 - \frac{\sqrt{2eV/\mu}}{v} \right)^2 dv. \tag{1.33}
\]

Differentiating twice both parts of the Equation (1.33) with respect to the potential \(V\), we obtain the relation:

\[
\frac{i''_+(V)}{enS_z} = -\frac{1}{8} \frac{\sqrt{-eV/\mu}}{V^2} \int_{-\infty}^{\infty} f(v) dv. \tag{1.34}
\]

Solving the Equation (1.34) for integral in its right hand part and differentiating this integral with respect to the variable \(V\), we obtain the Equation

\[
f\left(\sqrt{-2eV/\mu}\right) = 2 \frac{1}{\sqrt{\nu}} \frac{\partial}{\partial V} \left( V^{3/2} i''_+(V) \right) \tag{1.35}
\]

enabling one to deduce the electron distribution function over velocity \(f\left(\sqrt{-2eV/\mu}\right)\) from the second \(i''_+(V)\) and the third \(i'''_+(V)\) derivatives of the probe I-V characteristic at \(V \geq 0\).

It is easy to recognize that the value of the second term in the right-hand part of Eq. (1.33) is decreased rapidly to zero at \(V \to \infty\) for any reasonable distribution function \(f(v)\). To clear up this behavior of \(i_+(V)\), one can substitute the Maxwellian distribution function (see Eq. (1.9)) in the second term of the Equation (1.33) right-hand part. After this substitution and opening brackets under the integral, one can recognize three integrals for calculation:

\[
\text{Int}_1 = \frac{1}{4} \int_{-\infty}^{\infty} f(v) v dv = \frac{\nu_p}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \nu / \nu_p \right)^2 e^{-\left(\nu / \nu_p\right)^2} d\left(\left(\nu/\nu_p\right)^2\right) = \left[ \text{at } x = \left(\sqrt{-2eV/\mu}/\nu_p\right)^2 \right] = \frac{\nu_p}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{\nu}{\nu_p} \right)^2 \frac{\nu^x e^{-\nu}}{\nu_p} d\nu
\]
\[
\frac{v_{p}}{2\sqrt{\pi}} \left[ -xe^{-\frac{x}{\sqrt{2 \mu \left(\frac{2eV}{\mu} / v_{p}\right)}}} + \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} e^{-\frac{x}{\sqrt{2 \mu \left(\frac{2eV}{\mu} / v_{p}\right)}}} dx \right]
\]
\[
\frac{v_{p}}{2\sqrt{\pi}} \left( \frac{\sqrt{2eV / \mu}}{\sqrt{\pi}} \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} (v / v_{p}) e^{(v/v_{p})^{2}} d\left((v / v_{p})^{2}\right) \right)
\]
\[
\left. \frac{\sqrt{2eV / \mu}}{\sqrt{\pi}} \left[ -xe^{-x^{2}} \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} e^{x^{2}} dx \right] \right|_{x=\frac{2eV}{\mu} / v_{p}}^{\infty} + \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} e^{-x^{2}} dx
\]
\[
\frac{v_{p}}{\sqrt{\pi}} \left( \frac{\sqrt{2eV / \mu} / v_{p}}{\sqrt{\pi}} \right)^{2} e^{-\left(\frac{2eV / \mu}{v_{p}}\right)^{2}} + \frac{1}{2} \sqrt{2eV / \mu - \frac{1}{2} \sqrt{2eV / \mu} \text{erf}\left(\frac{2eV / \mu}{v_{p}}\right)};
\]
\[
\int_{\frac{2eV}{\mu} / v_{p}}^{\infty} \frac{f(v)}{v} dv = \left(\frac{\sqrt{2eV / \mu}}{\sqrt{\pi}} \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} \frac{v}{v_{p}^{3}} e^{-\left(\frac{2eV / \mu}{v_{p}}\right)^{2}} dv \right)
\]
\[
\left. \frac{v_{p}}{2\sqrt{\pi}} \left( \frac{\sqrt{2eV / \mu} / v_{p}}{\sqrt{\pi}} \right)^{2} e^{-\left(\frac{2eV / \mu}{v_{p}}\right)^{2}} \right|_{x=\frac{2eV}{\mu} / v_{p}}^{\infty} + \int_{\frac{2eV}{\mu} / v_{p}}^{\infty} e^{-x^{2}} dx
\]

In the process of \(\text{Int}_1\) and \(\text{Int}_2\) calculation, the expressions in the square brackets that appeared first in lines were obtained by the standard integrating by parts. For \(\text{Int}_2\) calculation, the equations

\[
\int_{0}^{\infty} e^{-x^{2}} dx = \frac{\sqrt{\pi}}{2}
\]

and

\[
\int_{0}^{\frac{2eV}{\mu} / v_{p}} e^{-x^{2}} dx = \frac{\sqrt{\pi}}{2} \text{erf}\left(\frac{2eV / \mu}{v_{p}}\right)
\]

were taken into account, where \(\text{erf}(z)\) is the error function. Substituting a composition \(\text{Int}_1 - \text{Int}_2 + \text{Int}_3\) in Eq. (1.33), performing necessary cancellations, and taking into consideration that \(v_{p} = \sqrt{2kT / \mu}\) for Maxwellian distribution function by definition, one can obtain the exclusively transparent final expression:
The form presented by Eq. (1.36) allows one to recognize contributions of its members in the probe current. In particular, it can be seen that the of charged particles concentration $n_{sq}$ in the probe vicinity can be estimated reliably by the expression:

$$\frac{i_{+M}(V)}{enS_z} \approx \frac{1}{2}\sqrt{-2eV/\mu} \, \text{erf}\left(\sqrt{-eV/kT}\right) + \frac{\langle v \rangle}{4} \exp\left(-\frac{eV}{kT}\right).$$

(1.36)

which follows directly from Eq. (1.36) at $eV > kT$. Indeed dividing Eq. (1.36) over Eq. (1.37) and taking into account that $\langle v \rangle = (2/\pi)^{1/2} kT/\mu$ for the Maxwellian distribution function, we obtain the ratio

$$\frac{n_{sq}}{n} = \frac{2i_{+M}}{enS_z \sqrt{-2eV/\mu}} = \text{erf}\left(\sqrt{-eV/kT}\right) + \frac{1}{\sqrt{\pi}} \sqrt{-kT/eV} \exp\left(-\frac{eV}{kT}\right).$$

(1.38)

The fragment of the dependence $n_{sq}/n = \varphi(eV/kT)$ expressed by the Equation (1.38) is plotted in Fig. 5 as the solid curve.

![Figure 5](image_url)