A High School First Course in

EUCLIDEAN PLANE
GEOMETRY

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The shape, position and
order are not basic but are
born in the imagination

Democritus

INTRODUCTION

The book is designed to promote the art and the skills of developing logical proofs of geometry propositions. It is concise, to the point and is presented to form a first course of geometry at high school level.

The content of the book is based on Euclid's five postulates and the most common theorems of plane geometry. Some of the theorems are introduced with detailed proofs. Other theorems are introduced because of their usefulness but their proofs are left as challenging problems to the users.

Problem solving examples are scattered throughout the book. They illustrate the organization of the proofs' written statements. Numerous annotations are introduced in the statements of the solutions to help the student understand the justification of the steps in the progress of the solution. The Author recommends that the student reproduce the solutions on his own to maximize the benefit from the book.

The content of the book is introduced in eleven chapters. The first chapter presents the five Euclid's postulates of plane geometry. The other chapters are organized in groups of subjects: the line, angles, triangles, quadrilaterals, similitude, circle, and elements of space geometry. A last chapter introduces strategies in calculating surface areas and volumes of non-simple objects. Those are mini engineering problems.

Each chapter is appended by a set of *Practice Problems* that could be worked out as class assignments or just for practice. Those problems are presented in three groups: construction problems, computational problems, and theorematical problems.

The first group helps the student to develop dexterity of drawing geometric figures using standard drawing instruments, a ruler, a protractor and a compass. The second group requires basic algebra skills. The student computes parts of geometric figures using data of other parts and pertinent theorems. Many of those problems are simplified versions of real engineering problems. They pave the way to workout the problems of the last chapters. The last group is where the student sharpens his talent of developing logical proofs.

Although the book is intended to be on plane geometry, the chapter on space geometry seems unavoidable. It helps understanding the figures and the shapes of solid objects. The surface area of solid objects can be computed using plane geometry formulae upon converting the surface of a solid to a plane surface. The portion of space comprised within the surface of a solid is a *volume*. Methods of calculating volumes of non-simple solids are introduced in that chapter.

The book has two companions, the Solutions Manual and the Power Point package. The users will have to acquire those separately. The Solutions Manual contains the detailed solutions to all the problems in the book. The Power Point package (incomplete at the time of printing this volume) contains animated and annotated presentations of geometric constructions. Some of those are described in the textbook; the others are answers to the construction problems presented at the end of each chapter of the book.

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PREFACE

This prologue is intended to share with the users my experience in teaching geometry using this book. The students who followed my recommendations about how to study with this book achieved a stunning improvement in their grades as well as in their initiatives in solving problems. This is how it worked with my students.

I require that all the students in my class have a binder dedicated exclusively to the geometry course and use the paper from that binder for the homework and for the quizzes. That way they will be able to re-file the graded papers in the same folder for future references. Also, I ask that the students use the binder to take notes of everything I write on the board. Most of them do!

In a typical new lesson I write the statement of a theorem and I draw the figure on the board. I carry out the proof and I ask the students to do the same on their binders; alternatively I start with a problem from the book. Then I present a new problem. I draw a figure on the board and I ask the students to prove something about it before I carry out the solution on the board. The new question could be to prove the theorem itself, but now using a different figure. The new figure could be the one I used in the proof of the theorem but now it is rotated or stretched and with different letters.

Sometimes I add lines to the new figure that are not relevant to the question. In those instances I instruct the students to focus on the parts of the figure that are relevant to the question and to ignore the non-relevant parts. I show them how to strip the extraneous information from the figure using problems from the book. An extreme example to this approach is illustrated in problem 26 of Chapter 6 in the Solutions Manual.

In some problems breaking up the figure into simpler ones is not an option. I highlight the parts that would make it easier for the students to see what in the figure is relevant to the question. A typical example to this approach is illustrated in problem 8 of Chapter 4 in the Solutions Manual.

The first requirement to do well in any course is to study. Almost everyday I assign a part from the book to be a quiz for the next day. The assignment is typically a theorem and its proof; alternatively the assignment could be a problem from the book. I instruct the students how to study; sometimes I ask the parents who care to help their children to manage the studying time at home and to make sure their children follow my instructions of studying.

I recommend that the students copy the assigned study theorem and its proof three times. I recommend not looking at the figure of the book while copying the theorem and the proof. I instruct the students to copy the figure on their papers. It is important to imprint the image of the figure and the location of the letters in the figure in the mind of the students. The letters of the figures identify the parts of the logical development of the proof. I encourage the students to use different letters from the ones used in the figure of the book.

I instruct the students to focus on memorizing the statement of the theorems as written; they will be able to state the same using their own language later. Also, I ask the students to memorize the steps and the structure of the proofs of the theorems. That's not harmful for a start. With practice on solving problems guided by the logical structure of the proofs they have studied, they will be able to develop their own logical proofs later. This has proven particularly true in problems that could be proven with different methods using different theorems. The memorization here is comparable to the memorization of the best move for a given disposition of the pieces on the chess game board.

After copying the assignment three times from the book, I recommend the students close the book and to try to redo the proof out of their memory, by first sketching the figure then writing the proof. After completing this rehearsal they should compare their proofs with that of the book. I recommend that the student fix their errors by writing them down on that last attempt of writing their proofs.

I ask the students to write the theorem and the proof without looking at the book for the last time, just before they go to bed. Again fix the errors by writing them on the paper of their solutions. By completing this task they should be ready for a successful next day quiz. It didn't take too long with my students before they begin to develop their own proofs without that lengthy rehearsal.

I instruct the students to do freehand sketches of the figures as closely as possible to the figure of the book. Most of my students had a hard time performing this task early in the course. Most of them improved on that deficiency. I do freehand sketches on the board and I ask the students to follow me step by step by drawing one line at a time on their papers with me. I rarely use drawing instruments except in geometric construction problems and I ask the students to follow my example. Most of them do!

I always use colors in sketching figures on the board. I observed that most students find it easier to distinguish the different parts of the colored figures. That helped improve their understanding of the question, which is essential for succeeding in carrying out the solution. Methods of solving problems and various alternatives to name angles and analyzing figures in problem solving are presented with ample details in the Solutions Manual.

I encourage the users to be creative.

TABLES OF SYMBOLS USED IN THE BOOK

Greek alphabet Highlighted lower case letters are used for angles		
α	A	**Alpha**
β	B	**Beta**
γ	Γ	**Gamma**
δ	Δ	**Delta**
ε	E	**Epsilon**
ζ	Z	Dzeta
η	H	**Eta**
θ	Θ	**Theta**
ι	I	Iota
κ	K	Kappa
λ	Λ	**Lambda**
μ	M	Mu
ν	N	Nu
ξ	Ξ	Ksi
ο	O	Omicron
π	Π	Pi
ρ	P	Rho
σ	Σ	**Sigma**
τ	T	Tau
υ	Y	Upsilon
φ	Φ	Phi
χ	X	Khi
ψ	Ψ	Psi
ω	Ω	**Omega**

Mathematical symbols used in this book
They are also defined at the first occurrence in the book

Symbol	Name	Description
$\parallel$	Parallel	Line *AB* parallel to line *CD*: $AB \parallel CD$
$\perp$	Perpendicular	Line *AB* *perpendicular* to line *CD*: $AB \perp CD$
$\cong$	Congruence	Object *A* congruent to object *B*: $A \cong B$
$\neg$	Negation	Object *A* is not object *B*: $A \neg B$ Read as: *A* not *B*.
AND OR NOT	Boolean logical operators	Association: both *A* and *B* together: *A* AND *B* Alternate: either *A* or *B*: *B*: *A* OR *B* Negation: one not the other: *A* NOT *B*
$\Rightarrow$	Implication	Also used in the book for **therefore**
$\Leftrightarrow$	Equivalence	Statement *A* equivalent to *B*: $A \Leftrightarrow B$
$\equiv$	Identity, definition	*A* is identical to *A*; also *A* is define as *B*: $A \equiv B$
$\exists$	Existence	There exists an object *Q*: $\exists Q$
$\bigcap$	Intersection	Line *m* intersects with line *k* at point *P*: $P \equiv m \bigcap k$ Read as: *P* is *m* inter *k*; also *P* is defined as
$\in$	Belongs to	$A \in k$: *A* is a point of line *k*; also *A* belongs to line *k*, *A* is contained in *k*.
$\widehat{A}$	Angle *A*	*A* is the vertex of the angle
$\widehat{BAD}$	Angle *A*	*A* is the vertex of the angle, *BA* and *AD* are its sides: read as angle *BAD*.
$\overset{\frown}{RB}$	Arc *RB*	*R* and *B* are the endpoints of the arc
$\overset{\frown}{ABC}$	Arc *ABC*	*A* and *C* are the endpoints of the arc, *B* is any point on the arc.
ABC	Triangle *ABC*	This symbol is used in this book only
$\propto$	Proportion	*a* is proportional to *b*: $a \propto b$
$\sim$	Similarity	*A* is similar to *B*: $A \sim B$

Annotations: annotations with an arrow, such as ← *use supplementary angles,* are inserted in the solutions of the examples in the body of the chapters to focus the attention of the student to what was done at that point of the solution. The student does not need to add those comments to his work.

An Asterisk **(*)** preceding the statement of the problem in the **Practice Problems** sections means there is a figure associated with the problem on that page.

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CHAPTER ONE

THE POSTULATES OF PLANE GEOMETRY

I. Euclid's Five Postulates

Geometry is a branch of mathematics that deals with shapes of objects drawn as lines, triangles, circles and boxes, among others. It is based on the definition of a ***point*** and some ***postulates***. A postulate is a proposition that is accepted true without proof. If a proposition could be proven then it is not a postulate.

Definitions:

- A **point** is something that has no part.
- A **plane** is a flat surface, such as the surface of the board. A plane has no thickness.

We visualize a point by a spot on the plane. A practical plane would be a flat sheet of paper, or the board. The spot contains an infinite number of points. However, for all practical purposes a point is always visualized by marks of small shapes. We usually label a point by an upper case letter, such as point *A* (fig. 1). The figures we work with in this course are drawn in a plane, such as the sheet of paper, or the board. For this reason our geometry is said ***plane geometry***.

Figure 1. A point.

Euclidean plane geometry is based on five Postulates all of which make use of the definition of a point and a plane.

Postulate 1. Only one straight-line segment can be drawn joining any two points.

A ***straight line segment***, or simply a ***segment*** that joins two points *A* and *B* may be visualized by marking the trace of a pencil on the paper with the help of a ruler. A line segment is named by its endpoints, such as segment *AB* (fig. 2). We could write *AB* or *BA*. It is the same segment in this book.

Figure 2. A line segment AB.

We note *en passant* that a *directed* segment is presented by its endpoints letters and a bar atop of the letters, such as $\overline{AB}$, which means: the segment stretches from the starting point A to the end point B. Therefore, $\overline{AB}$ and $\overline{BA}$ are two different segments. Directed segments are not covered in this book.

Postulate 2. A straight-line segment can be extended indefinitely on either sides in a straight line.

A straight line is always of an infinite length. It is impossible to visualize the entire line. Instead we draw a segment of a straight-line without specifying endpoints. Such a straight line is referred to as a line through points A and B (fig. 3). Most often we do not need to use two points to specify a line. We draw a line segment without endpoints and we label it by a lower case letter and we say simply ***line a***. Such a segment is always understood as a line of infinite length.

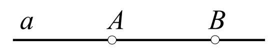

Figure 3. A line of infinite length.

We could mark any number of points on a line. Points on a straight-line are said ***collinear***, e.g. points A and B of figure 3 are collinear. You may add many more points in addition to A and B to line a of figure 3. They are collinear.

Postulate 3. Given a segment of a straight-line, a circle can be drawn having the segment as a radius and one endpoint as a center.

This postulate expresses the association of a line segment with a circle so that one endpoint of the segment is the ***center*** O of the circle and the other endpoint A is on the circle; the segment OA is the ***radius*** of the circle (fig. 4). It is a practice to label a circle by the letter of its center, such as circle O.

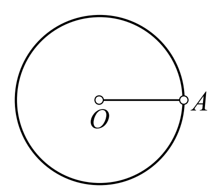

Figure 4. A circle

By this postulate the circle is the ***locus*** of points that can be found in the plane at a fixed distance from a fixed point O called the center of the circle.

***Definition*:**

If each part of one object coincides exactly with its corresponding part of another object, the two objects are said **congruent**

(`congruence` *is to objects as* `equal` *is to numbers.*)

Postulate 4. All right angles are congruent.

A right angle is a shape of an object in the plane. We visualize the shape of a right angle by two lines drawn through the same point and a square figure around the point as illustrated in figure 5. Think of a right angle as a shape of an object not as a number of degrees.

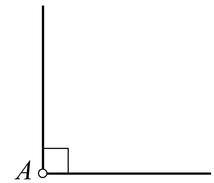

Figure 5. A right angle

Angles in general are not congruent but right angles are always congruent.

Postulate 5. Given a straight line and a point off the line, there exists only one line through that point that never intersects with the given line no matter how far we extend the two lines.

We will encounter a variation to the statement of the fifth postulate but it always expresses the same thing: two lines, such as *m* and *n* of figure 6, are ***parallel*** if they never intersect. Sometimes we say: two lines that intersect at infinity are parallel. Using symbolic notation we write: $m \parallel n$ (read *m* parallel to *n*).

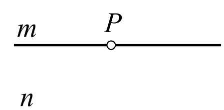

Figure 6. Two parallel lines

Types of lines

A line is always understood to be a *straight line* unless the type of the line is specified. There are three different types of lines shown in figure 7. The *broken* line is made of segments of lines attached by their endpoints and each segment differs by its length and its direction from the sides segments. A *curve* line is continuously changing in direction.

The length of a line extends indefinitely on either side. It is not limited to the drawn portion of the line.

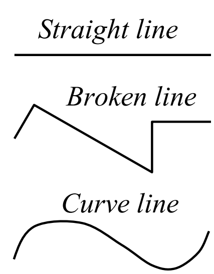

Figure 7. Types of lines

The unit of length

In a practical world a line segment is a ***distance*** between two points, which are the endpoints of the segment. The distance between the endpoints of a segment is also called the ***length***, or the ***measure*** of the segment.

The unit of length of lines is the ***meter***, it is dived into subunits: the ***centimeter*** and the ***millimeter***, among others, are commonly used in this book. The symbols of units of length are written as lower case characters as follows:

m for meter,
cm for centimeter and
mm for millimeter.

The correspondence between the meter and its subunits is as follows:

$$1 \text{ m} = 100 \text{ cm} = 10^2 \text{ cm}$$
$$1 \text{ cm} = 10 \text{ mm}$$
$$1 \text{ m} = 1000 \text{ mm} = 10^3 \text{ mm}$$

Other units of length inherited from the old British system of units are still used as practical units of length in the United States. The meter is the legal unit of length all over the world including the United States.

II. Geometric construction of segments

In order to perform geometric operations with segments of lines we need to know how to draw lines of the same length, parallel lines, and lines that form a right angle. These operations are ***geometric construction***. To do geometric constructions we need a ruler and a compass. A ruler is a straight edge marked in centimeters and millimeters.

Construction of a segment of a given length

Compass method

Construct a segment *PQ* of the same length of a given segment *AB* (fig. 8).

- Mark a point *P* on the paper.
- Place the pin of the compass at point *A* and open the compass so that the tip of the pencil is at point *B*.
- Without changing the opening of the compass, place the pin at *P* and draw an arc.
- Mark a point *Q* on the arc and draw the segment *PQ*.
- Segment *PQ* has the same length as *AB*.

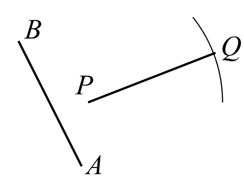

Figure 8

Ruler method

If the measure of segment *AB* is specified in units of length, say 5 cm, draw a line with the ruler and mark a point *P* on the line. Use the ruler and mark another point *Q* on the line 5 cm from *P*. This method is less rigorous than the compass method but it is practical and is justified by its simplicity. *Engineering*

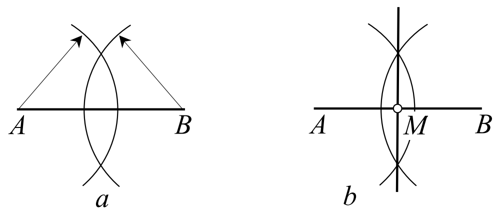

Figure 9. Construction of the midpoint of a segment.

graph papers are graduated in centimeters and millimeters and can be used for construction as well.

Construction of the midpoint of segment AB

Compass method

- Open the compass a little longer than half of the segment length. Place the pin of the compass at point *A* and draw an arc that extends to either sides of *AB* (fig. 9*a*).
- Without changing the opening of the compass place the pin at point *B* and draw an arc that intersects with the first arc at either side of *AB* (fig. 9*a*).
- Join the intersection points of the two arcs with a line that intersects with *AB* at *M* (fig. 9*b*).
- Point *M* is the midpoint of segment *AB*.

Ruler method

Use the ruler to measure the length of the segment. Compute half the length and mark a point on the segment at a distance from one endpoint equal to the half-length you just computed. Label the new point by a letter, say *M*. Point *M* is the midpoint of the segment. This method is less rigorous than the compass method but it is practical and is justified by its simplicity.

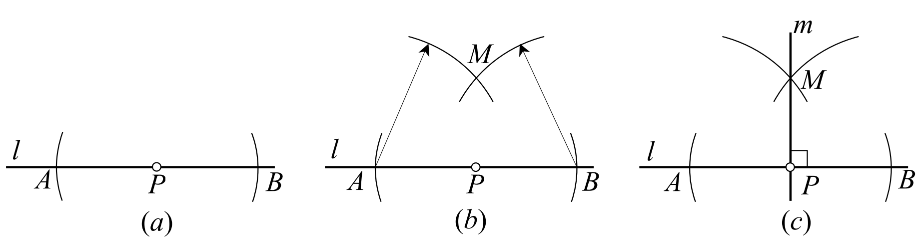

Figure 10. Construction of a line *m* at right angle with line *l* at point *P*.