Geometry by Construction

Geometry by Construction

Object Creation and Problem-solving in Euclidean and Non-Euclidean Geometries

Michael McDaniel

Universal-Publishers
Boca Raton

Geometry by Construction:
Object Creation and Problem-solving in Euclidean and Non-Euclidean Geometries

Universal-Publishers
Boca Raton, Florida
USA • 2015

ISBN-10: 1-62734-028-9
ISBN-13: 978-1-62734-028-1

www.universal-publishers.com

Cover image: Kamira/Bigstock.com

Publisher's Cataloging-in-Publication Data

McDaniel, Michael.
Geometry by construction : object creation and problem-solving in euclidean and non-euclidean geometries / Michael McDaniel.
pages cm
Includes bibliographical references and index.
ISBN: 978-1-62734-028-1 (pbk.)
1. Euclid's Elements. 2. Geometry, Non-Euclidean. 3. Geometry, Modern. 4. Geometry—Foundations. 5. Geometry—Problems, Exercises, etc. I. Title.
QA445 .M34 2015
516—dc23

2015930041

Contents

Michael McDaniel
Aquinas College

To Dennison and Marguerite Mohler and to Ted Thompson: your vision has inspired new and interesting geometry. Thank you for the opportunities!

Preface

In 1919, a solar eclipse allowed British researchers to photograph stars whose light had to pass the sun before reaching Earth. Twelve years earlier, Einstein had predicted the sun's mass would bend the fabric of space, making such a star appear out of place in the sky. He had even provided the formulas to determine how much the star's position would be misplaced. When the photographic plates were developed and measurements taken, the star matched his prediction and the universe became more clearly understood. The planets are not billiard balls, spinning and rotating around each other in a box, their movements governed by their gravitational attraction; the box is more like a trampoline and the stars and planets bend the trampoline.

As a consequence, we know the universe does not fit Euclidean geometry. The geometry which has served us so well on Earth all these years is not exactly correct at the celestial level. Although we cannot hitch rides with angels and perceive our universe from some point outside it, we can play with models of bent geometries, which is exactly what we will do in this book. In fact, we will look down on the bent spaces and see them in their entireties. We will be outside the bent spaces, working in a Euclidean space which contains these bent spaces.

Using only the most ancient of drawing tools, the compass and straightedge, we will make the two-dimensional objects which might illustrate the essential structure of the universe. For those of us with access to a dynamic geometry program, like *Geometer's SketchPad*, we can bring the ancient tools into our century of animations, precise measurements and color printing.

The only way to fully appreciate the geometrical worlds ahead is with a notebook, compass and straightedge handy. Prepare to keep your hands busy.

On the philosophy of the book

We focus on constructions for some good reasons. We get to make and see what we are studying. We also experience geometry as a living discipline because this book contains some new results from undergraduate research and a few new ideas from the author which tie the content together. For example, this book contains the constructions for hyperbolic lines and elliptic lines which are a single reflection away from being the same construction. Some texts have results from living mathematicians; here we have new, verified results written when the authors were students.

In this way, the text has been written as a door to a hallway to another, more important door: here is an introduction to actual, publishable undergraduate research. Down the hall of undergraduate research, we find the satisfactions and challenges of full-time mathematical work. It is the author's greatest hope that this book helps a few more students through those doors.

Throughout the development of the theorems and constructions in this book, the student contributors and the author noticed that every time proof was sought for a solid conjecture, there was always enough material to demonstrate the truth. We waded into the unknown, heading toward the desired result and every time, what we needed existed. The search put professor and students on the same footing: none of us knew how to find the answer! We developed confidence and comfort in the generosity of mathematics. Beyond the richness of the subject, we often felt as some philosophers have written, the Truth was there, waiting to be discovered. More than waiting – an active give and take, sparring, wrestling took place as we sought the support we needed for the theorems.

Compare this experience with the mundane, disappointingly common pastime of playing Solitaire. How many times do we go through the deck, never finding one of the cards we need to continue the game? The majority of times, we lose. But in math, when we seek the truth, we almost always find what we need to prove that truth. In fact, when a problem eludes proof for a long time or ends up as one of the unprovable ideas, that problem gets famous!

Whether the reader believes this or not, our advice might be worth taking. When the Truth is talking, listen and take advantage. Trust that you are strong enough for the intellectual battles and that the math itself wants to help! The successful reader will find the unwanted terms cancel out, or the angles we need congruent lie in similar triangles, or whatever it takes to force the true conclusion; the Truth is in reach.

Geometry by Construction

Chapter 1

Euclidean geometry rules and constructions

Since our two-dimensional versions of bent space require Euclidean geometry, we will start with that geometry. Here's a brief review of the geometry of Pythagoras, Newton and Kant: Euclidean geometry. When the time comes, we will see that the bent geometries have the same logical standing as Euclidean. In fact, we will study the bent spaces in terms of Euclidean, sort of like astronauts who work in orbit: some Earthly rules apply while others do not apply.

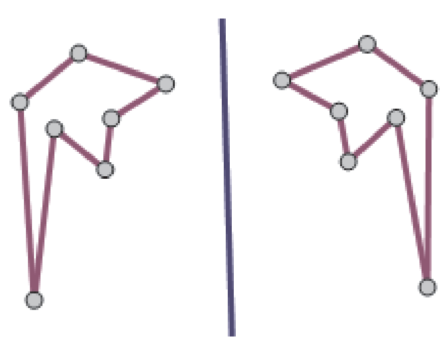
These pages of rules and constructions are absolutely essential knowledge for this book. We will explore many of those rules. We will also refer to rules as if they are familiar, even if they have not turned up in any previous problems.

1.1 Euclidean Geometry Vocabulary and Definitions

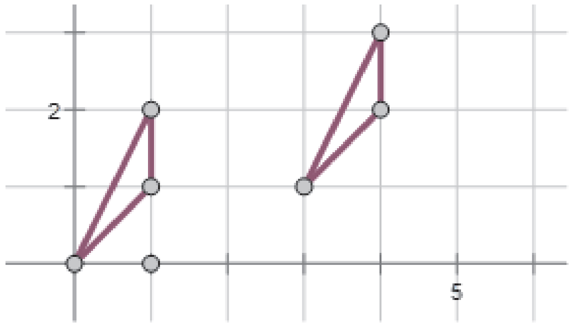
Learn these quickly and never get them mixed up.
Point: That which has no size; it has zero dimensions. A point is pure location.
Line: The set of points which lie between two distinct points and the points included when the segment is extended. When the segment is extended in only one direction, this makes a ray. Lines, segments and rays are one dimensional. Points on a single line are called collinear.
Plane: The set of all points on two intersecting lines, along with the points between any two points, one on each line. A plane is two dimensional. Points in the same plane are called coplanar.
Between: Point B lies between points A and C if the distance from A to B plus the distance from B to C equals the distance from A to C.

Isometries

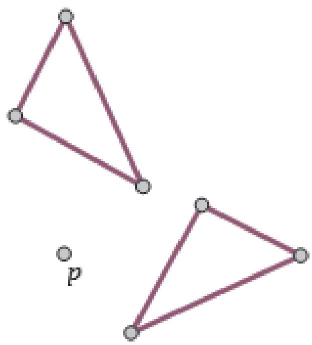
An isometry is a transformation which preserves distances and angle measures.
Reflection: Reflect across a line. The line acts as a mirror. The reflection sends each point to the point on the line perpendicular to the line of reflection, at the same distance from the line of reflection as the starting point. Usage: *Ref* l (object) where l is the line of reflection and object is the object being reflected. This is the only basic isometry which reverses orientation.



Translation: Move the object in the plane. All points have their x and y coordinates changed by the same two amounts. Usage: $f(x, y) = (a + x, b + y)$.



Rotation: Rotate an object around a point by an angle of a certain size. Usage: Rot(P, θ,object) where P is the point around which the rotation occurs and θ is the angle of rotation.



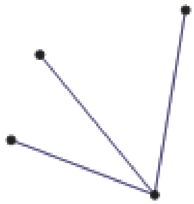
Special Lines

Notation: We will use the nouns segment, ray and line before two letters to indicate which object we mean. The length of the segment, AB, has no modifier and this indicates a number.
Parallel: Coplanar lines which have no intersection are parallel.
Perpendicular: Lines which meet at right angles are perpendicular.
Segment Bisector: A line containing the midpoint of a segment.
Angle Bisector: A line through the vertex of an angle which divides the angle into two congruent angles.
Angle: Two rays, a ray and a segment or two segments joined at an endpoint (called the vertex) form a set of points called an angle. We define 360 degrees as the rotation of one ray around its end point exactly once; so one degree is 1/360 of a complete rotation.

Special Angles

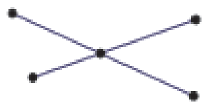
These angles get their names from their positions.

Adjacent: Two angles with the same vertex and one shared side, but with no interior points in



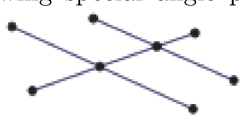
common.

Vertical: Two intersecting lines form two pairs of equal angles. Each pair is vertical. They form



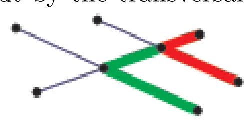
a letter X-shape.

The following special angle positions occur when two lines are crossed by a



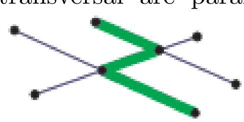
transversal.

Corresponding: This pair are in the same relative position, as in both being upper right. A pair of these form an F shape. The two lines cut by the transversal are



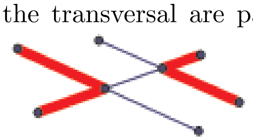
parallel if and only if corresponding angles are congruent.

Alternate Interior: Their sides form a Z-shape. The two lines cut by the transversal are parallel if and only if the alternate interior angles are equal.



Alternate Exterior: These angles are vertical with a pair of alternate interior angles.

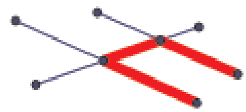
The two lines cut by the transversal are parallel if and only if the alternate



exterior angles are equal.

Same-side Interior: These angles form a block C-shape.

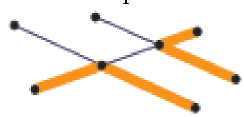
The two lines cut by the transversal are parallel if and only if the same-side



interior angles are supplementary.

Same-side Exterior: These angles are adjacent to a pair of same-side interior angles.

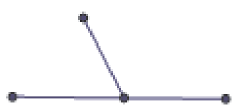
The two lines cut by the transversal are parallel if and only if the same-side



exterior angles are supplementary.

Supplementary: Two angles are supplementary when their angle measures add up to 180 degrees.

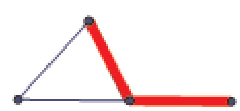
A special case of supplementary angles: Adjacent angles with collinear exterior sides are supplementary. These are sometimes called a linear pair.



Complementary: Two angles are complementary when their angle measures add up to 90 degrees.

A special case of complementary angles: Adjacent angles with perpendicular exterior sides are complementary.

External Angle: When one side of a polygon is extended, the angle formed by the extended side and next side of the polygon is an external angle. In a triangle, an external angle is equal to the sum of the remote interior angles (the two interior



angles which are not adjacent to the exterior angle.)

Straight Angle: An angle whose measure is 180 degrees is a straight angle. The exterior sides form a line.

Obtuse Angle: An angle whose measure is between 90 and 180 degrees.

Right angle: An angle whose measure is 90 degrees is a right angle.

Acute Angle: An angle whose measure is less than 90 degrees is an acute angle.

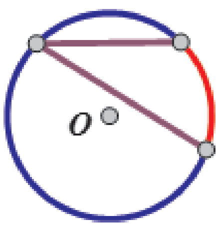
Bisectors

Perpendicular Bisector of a segment: A line, ray, or segment which is perpendicular to a segment at that segment's midpoint is a perpendicular bisector of a segment.

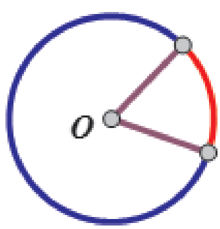
Angle Bisector: A line which cuts an angle into two equal angles is an angle bisector. Any point on an angle bisector is equidistant from the sides of the angle.

Circles

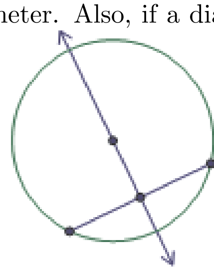
A circle is the set of all points equidistant from a center point.
Radius: A segment from the center to any point on the circle. All radii are congruent.
Chord: A segment whose endpoints are on a circle.
Diameter: The largest chord in a circle. A chord which contains the center.
Arc: The part of a circle between two points. We name arcs greater than or equal to 180 degrees with three letters.
Inscribed angle: An angle with its vertex on the circle and whose two sides are chords or a chord and a tangent. An inscribed angle equals half the measure of its



arc.
Tangent: A line which intersects a circle in exactly one point. The radius is perpendicular to the line at the point of tangency.
Secant: A line which intersects a circle in two points.
Central angle: An angle whose vertex is the center of the circle. The central angle



has a measure equal to its arc.
Sword Theorem: A diameter which is perpendicular to a chord bisects the chord and its two arcs. Alternatively, if a chord is the perpendicular bisector of another chord, then the bisector is a diameter. Also, if a diameter bisects a chord, then it



is perpendicular to the chord.
Concentric circles: Concentric circles have the same center and different radii lengths.

Triangles

A triangle consists of three non-collinear points (vertices) and the segments joining them.
Degenerate: If the three vertices are collinear, then the triangle is actually just a segment. Or all 3 points are the same point. Each case is referred to as a degenerate triangle.
Isosceles: An isosceles triangle has two sides congruent. The third side is called the base and the vertex opposite the base is The Vertex.
Acute: All angles in an acute triangle are less than 90 degrees.
Right: A right triangle has an angle of 90 degrees.
Obtuse: An obtuse triangle has an angle of more than 90 degrees.
Equilateral or **Equiangular**: A regular triangle: all sides and all angles are congruent.
Median: The segment from a vertex of a triangle to the midpoint of the opposite side is a median.
Altitude: The segment from a vertex perpendicular to the opposite side is an altitude.
Similar: If two pairs of angles of two triangles are congruent then the triangles are similar. Similar triangles have corresponding sides in proportion. So if triangle ABC is similar to triangle DEF, then $\frac{AB}{BC} = \frac{DE}{EF}$. An alternative method for showing two triangles are similar is by having a proportion of two pairs of sides and having the included angles congruent.
Congruent: A pair of congruent triangles have the same size and the same shape. Five ways to prove two triangles are congruent have acronyms based on the corresponding parts which match up.

SSS: If two triangles have three pairs of congruent sides, then the triangles must be congruent.

SAS: If two triangles have two pairs of congruent sides and the included angles are also congruent, the triangles are congruent.

ASA: If two triangles have two pairs of congruent angles and the included sides are congruent, the triangles are congruent.

AAS: If two triangles have two pairs of congruent angles and a corresponding pair of non-included sides are congruent, then the triangles are congruent.

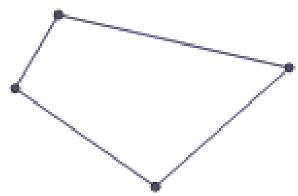
HL: If two right triangles have a pair of sides (legs) congruent and their hypotenuses are congruent, the triangles are congruent.

These five ways to prove triangles congruent are theorems, not axioms. The proofs rely on isometries.
The Sine Law: For any triangle ABC, the following proportions hold.
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

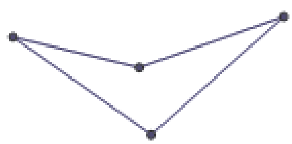
The Law of Cosines: For any triangle ABC, the following equation holds.
$a^2 = b^2 + c^2 - 2bc \cos A$.

Quadrilaterals

Convex: If a band is stretched around the convex quadrilateral, the band sticks to the sides with no gaps. Precisely, the segment between any two points on or in the



quadrilateral is in the quadrilateral.
Concave: If a band is stretched around the concave quadrilateral, there is a place where the band does not touch the quadrilateral. Precisely, there exist two points of the quadrilateral such that the segment between them contains points outside



the quadrilateral.
Cyclic: If a circle can contain all four vertices, the quadrilateral is cyclic.
Trapezoid: A quadrilateral with only one pair of opposite sides parallel is a trapezoid.
Square: A regular quadrilateral: all angles and sides are congruent.
Rhombus: An equilateral quadrilateral.
Rectangle: An equiangular quadrilateral.
Parallelogram: Opposite sides parallel.
Kite: A quadrilateral with two pairs of consecutive, congruent sides is a kite.

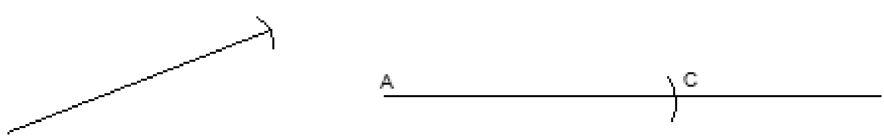
1.2 Constructions to know

The constructions which follow are for compass and straightedge. (*SketchPad* does these with a few clicks.) Some colored pens or pencils are useful for complicated drawings. Always place your drawing page on a stack of paper: this gives the point of your compass a place to dig in and stabilize. Also, choose a compass without a rivet holding the pivot together. A nut and bolt or screw can be tightened. (As you use your compass, the arms will get a little loose.) Being able to tighten your compass keeps your drawings accurate. Some compasses end with a narrow cylinder, able to hold only a standard pencil. By choosing a compass with an adjustable barrel, you can switch pens and pencils in order to keep your drawing organized by color.

When constructing, we almost never get to pick points out of thin air. One instance where we do get to start from nothing is beginning the construction. Even when we are given a length or angle to use, we will almost always have to copy these to our own work space. The initial placement of the first segment requires a bit of forethought: leave space for your work to happen. Do not begin near an edge of the paper and then build so that your work quickly lies outside the margin. Assume that your construction will take up a lot of space, so give yourself room and make your starting objects large enough to fit all the stuff you plan to do.

Copying Lengths and Angles

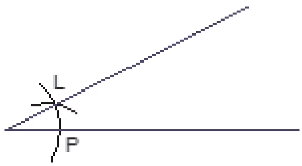
To copy a length, lay down a segment which is longer than the one you intend to copy. (The given segment is unlabeled.) Then place the point of your compass at one end of the given segment and adjust the radius so that your pencil tip meets the other endpoint. Make a little arc. Now move your compass to the workspace and place the compass point at one endpoint of your long segment (point A.) Mark an arc with your pencil on the segment. The arc intersects the segment at a point. Segment AC is congruent to given the segment.

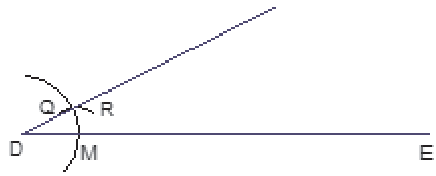


To copy an angle, again draw a segment DE in your workspace. Mark an arc across the angle you wish to copy and then put the point of your compass at endpoint D to mark an arc with the same radius, across the segment in your workspace. Go back to the first drawing and place the point of the compass at a point where the arc crosses a side of the angle. Adjust the radius so that the pencil lands on the other intersection of the arc with a side of the angle. You have just captured the width of a chord PL, which will help build a congruent angle. Mark a little arc in order to show what you have done.

Place your compass point at the intersection of the arc and segment in your workplace M. Draw an arc QR which intersects your first arc. Construct a segment through their intersection and the endpoint of the segment which you have used before, DK. The first segment and this segment form an angle congruent to the given angle.

It is usually the case that, to prove a construction works, we will have to draw segments which are otherwise invisible in our construction. It will also be good to remember that copied radii are congruent, even when the entire circles are not drawn.



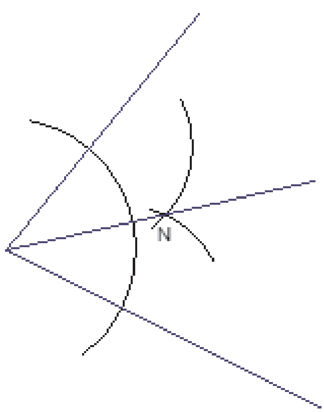


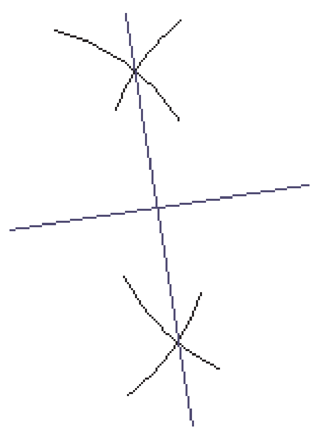
On paper, you must also leave behind marks which indicate how a construction was performed. Keep the extras to a minimum: do not draw arcs much longer then they need to be. Place construction marks well away from where the action is whenever possible, leaving room for future steps. A good construction avoids needless complexities like extra points of intersection.

Bisectors

We have two kinds of bisectors: angle bisectors and segment bisectors. In both cases a line does the actual bisecting: the bisector is named after what gets bisected. It might be said that the midpoint of a segment bisects the segment; but we will not say that here because the midpoint is a point on the segment; a bisector goes beyond the object being cut in half.

The angle bisector construction starts similar to copying an angle. This time, we draw arcs from both points where the first arc intersects the sides. The point N where the arcs intersect combines with the vertex of the given angle to define the angle bisector.

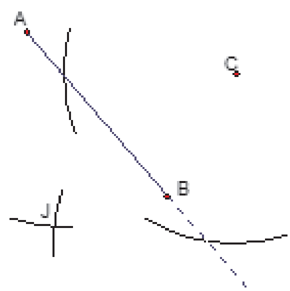




The perpendicular bisector of a segment is one of the easiest constructions we have. Simply choose the radius of your compass greater than half the length of the given segment. Place the point of the compass at each endpoint and draw the arcs long enough to intersect above and below the segment. The perpendicular bisector passes through the two points where the arcs intersect.

Perpendicular Segments

There are two more situations where we will construct specific perpendiculars: through a point on a line and through a point not on a line. Their constructions are virtually the same, so we will show the construction for a point not on a line. We are given the point C and the segment AB. Since the point C was so close to one endpoint, we extend the segment (the dashed part) in order to get some working room. The reader who has been doing the constructions all along can probably see that we made an arc which cut the extended segment twice. Then we used those two points of intersection as centers for two new arcs which intersected on the other side of the segment. The undrawn segment CJ is perpendicular to segment AB and it contains C, as required.

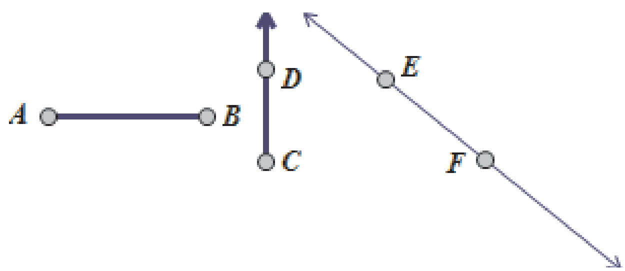


All the geometry we will do in this section is called Euclidean geometry because of its connections with the Greek mathematician Euclid. His Elements were volumes of geometrical knowledge which set the example for mathematical exploration: the author gives the assumptions and definitions and builds from there using logic. Many books exist which deduce geometry in exactly this way: they follow the precise reasoning which handles every detail of the development of Euclidean geometry. This work establishes geometry in ways which few subjects outside mathematics can match – assumptions made clear, all results are known to be true, no editorial agenda, no quest for big bucks.

We, however, will follow an easier path than a purely formal development. We will indeed follow the rules of logic and prove many of our theorems, but some results we will simply take as true and move on. We will rely on more than axioms, definitions and previous conclusions. We will construct the objects of our study using compass and straightedge so that we can see what we are talking about.

Everything we make in this course is made of points. A point has no size; it is pure location. The first Euclidean axiom assumes that two distinct points determine exactly one line. This is our first construction.

Write down two dots, the points A and B. Now take your ruler and draw the unique line segment through those two dots. If you write an arrow on the ends of the segment, you have constructed a line. We won't be drawing those little arrows at the ends very often. But we will have occasions where the accuracy of the notation allows us to explain our work with great precision. This notation is not meant to drive everybody crazy with pickiness. But there are times when we need to refer to a segment, a ray or a line so that every reader knows what we mean. So here are examples. We have pictured segment $\overline{AB}$, ray $\overrightarrow{CD}$ and line $\overleftrightarrow{EF}$. It would be incorrect to call the ray $\overrightarrow{DC}$ because, for the ray, we say the endpoint letter gets written first and endpoint letter gets the endpoint symbol.



This is the first of many notations to keep clear. The letters AB alone stand for the length, which is a number. The segment $\overline{AB}$ is a set of points, not a number. Such attention to detail, however, could get tedious in a rapid conversation. On those occasions, we will allow AB to stand for whatever it needs to stand for and, if the writing establishes context, that meaning should be clear. It is hoped this double standard resembles real life: everyday-talk is quick and easy, written work requires attention to detail.

The two arrows on the ends of a line stand for another Euclidean axiom – lines can be extended indefinitely. This is a useful assumption when two lines are constructed which do not meet on the paper; yet, theoretically, we have reason to believe they do in fact meet eventually. Then we don't have to tape another piece of paper to our first one in order to imagine their intersection.

The figures in this book were made in *SketchPad*. The lines there do not have the arrows drawn in; but the arrows do appear when lines are printed as part of a drawing. Constructing the line through two points is a crucial skill. One way is to click on the two points and go to the Construct menu. As long as only those two points are selected, the choices construct segment, construct line and construct ray will be enabled. The order in which the points were selected determines the way the ray points, just the same as the way we write them down. When two lines are constructed so that they intersect, select the two lines and construct intersection. *SketchPad* then extends the drawing so that the point of intersection exists in the drawing. The reader might have to scroll to see it; but the point of intersection is there.

The first axiom matches the way we construct: we very rarely get to draw a line just anywhere. Usually, we have two significant points present in the drawing; these determine a line. Then we place a pencil-tip at one point, hold the ruler next to the pencil and along the second point and we draw the desired line. As easy as it sounds, some care must be taken to get the line through the two points, not just near them.

Two points also determine a circle when one point is the center and the other is a point on the circle. (The existence of circles is the next Euclidean axiom.) In *SketchPad*, select the points in the order we want and the choice Construct circle by center and point is enabled under the Construct menu. Drawing a circle on paper is what a compass is built to do. First, get a few sheets of paper under your construction so your compass point has a foothold. Place the point at the center and put a little pressure on it as you draw the circle. Draw the circle once, nice and neat. Drawing it twice only makes the circle thicker and if the compass wiggles a little bit, the correct circle is lost in a whirl of errant arcs.

1.3 Tangent construction

Suppose we wanted to construct the light rays going from the sun to the Earth. Compared to the sun, the earth is a speck. So we'll treat the Earth as point A and the sun as circle O. We are going to need tangents to circle O through point A because the rays going from the sun to Earth are the only rays we want. Any other light rays cannot or will not hit point A. We are not allowed to simply lay down the straightedge so that we draw a line through A that appears tangent to circle O. We must find a point of tangency and lay down the edge along A and the point of tangency: two points determine a line. Precision will get us safely through the bent spaces and that precision starts now. Stated as a geometry question, we have the following given.

Problem. Given a circle with center O and a point A outside the circle, construct the two lines through A tangent to the circle.