R² - Heaps with Suspended Relaxation for Manipulating Priority Queues and a New Algorithm for Reweighting Graphs

by

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R²-HEAPS WITH SUSPENDED RELAXATION FOR MANIPULATING
PRIORITY QUEUES

AND

A NEW ALGORITHM FOR REWEIGHTING GRAPHS

by

RUTH SHRAIRMAN

A thesis submitted to the
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This research is dedicated to two main problems in finding shortest paths in the graphs. The first problem is to find shortest paths from an origin to all other vertices in non-negatively weighted graph. The second problem is the same, except it is allowed that some edges are negative. This is a more difficult problem that can be solved by relatively complicated algorithms.

We attack the first problem by introducing a new data structure - Relaxed Heaps, that implements efficiently two main operations critical for the improvement of Dijkstra's shortest path algorithm. \(R^2\)-heaps with suspended relaxation proposed in this research gives the best known worst-case time bounds of \(O(1)\) for a decrease_key operation and \(O(\log n)\) for a delete_min operation. That results in the best worst-case running time for Dijkstra's algorithm \(O(m+n\log n)\) and efficient in practical implementation.

The empirical study with \(R^2\)-heaps demonstrated strong advantage of its use for Dijkstra's algorithm over the "raw" Dijkstra's without heaps. This advantage is especially dramatic.

\(R^2\)-heaps can be used in a large number of applications in which set manipulations should be implemented efficiently.

For the problem of finding shortest paths in graphs with some negative edges, we present a new approach of reweighting graphs by first reducing the graph to its "canonical" form, which allows to apply an effective algorithm to reweight the graph to one with non-negative edges only and simultaneously to find shortest paths from an origin to all other vertices in the graph. This approach allows to give new algebraic and geometric interpretations of the problem. The experiment with the Sweeping Algorithm demonstrated \(O(n^2 \log n)\) expected time complexity.

These results open new prospects to improve algorithms for a wide variety of problems including different network optimization problems that use Dijkstra's algorithm as a subroutine, as well as multiple Operations Research and Modeling problems that can be reduced to finding shortest paths on graphs.
DEDICATIONS

To the memory of my mother, Riva
whose boundless love and firm belief in me
helped me to accomplish my goals

and

To my family,
Alexander, Igor, and Daniel
for their love and support
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CONTENTS

Part I. $R^2$-heaps With Suspended Relaxation For Manipulating Priority Queues

CHAPTER 1 INTRODUCTION

1.1 Review of previous work 1
1.2 Definitions and notations 7

CHAPTER 2 RELAXED HEAPS

2.1 Definitions 10
2.2 Rank relaxed heaps 12
2.2.2 Transformations 13
2.2.3 Delete_min operation 18
2.2.4 Correctness and complexity 18
2.2.5 Other operations 19
2.3 Run relaxed heaps 20

CHAPTER 3 $R^2$-HEAPS WITH SUSPENDED RELAXATION

3.1 The main concepts of $R^2$-heaps organization 26
3.2 Architectural design of $R^2$-heaps 28
3.3 Representation of data in $R^2$-heaps 31
3.4 Decrease_key operation 34
3.4.1 Decrease_key algorithm 35
3.4.2 The transformations 37
3.5 Delete_min operation 47
3.5.1 Find_min algorithm 47
3.5.2 Remove_min algorithm

CHAPTER 4 IMPLEMENTATION OF DIJKSTRA'S ALGORITHM

4.1 Dijkstra's shortest path algorithm

4.1.1 A sequential implementation

4.1.2 A parallel implementation

CHAPTER 5 EMPIRICAL STUDY OF THE EFFICIENCY OF R\(^2\)-HEAPS

5.1 General

5.2 Organization of the experiment

5.3 Implementation of Dijkstra's algorithm with use of R\(^2\) heaps

5.4 Analysis of the empirical data

5.4.1 Complete graphs with use of R\(^2\)-heaps

5.4.2 Sparse graphs with use of R\(^2\)-heaps

5.4.3 Comparison of Dijkstra's algorithm implementations with and without R\(^2\)-heaps

CHAPTER 6 CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

6.2 Future work

Part II. A New Algorithm for Reweighting Graphs

CHAPTER 7 INTRODUCTION

7.1 The state-of-the-art

7.2 Definitions and notations
7.3 Hamiltonian CCT 85

7.4 Canonical matrix 86

CHAPTER 8 ALGEBRAIC REPRESENTATION OF THE PROBLEM

8.1 Decomposition of vector V 90

8.2 System of double linear inequalities 90

8.3 Solving the system by the Fourier method 93

CHAPTER 9 GEOMETRIC REPRESENTATION OF THE PROBLEM

9.1 Convex polytope 99

CHAPTER 10 REWEIGHTING GRAPHS AND SHORTEST PATHS ALGORITHMS

10.1 Sweeping algorithm 102

10.2 Interrelation between different representations of the problem 105

10.3 Proof of correctness and convergency of the sweeping procedure 107

10.4 Algorithm for all-pairs shortest paths 116

CHAPTER 11 EMPIRICAL STUDY

11.1 Random graphs 118

11.2 Organization of the experiment 119

11.3 Discussion of the experimental results 121

CHAPTER 12 CONCLUSIONS AND FUTURE WORK

12.1 Conclusions 124

12.2 Future research 125

BIBLIOGRAPHY 129
TABLES

Table

1. Factors for \( m \) and \( n \log n \) in time bound for the complete graphs in Dijkstra's algorithm with R²-heaps 63

2. Factors for \( m \) and \( n \log n \) in time bound of Dijkstra's algorithm with R²-heaps for sparse graphs of 500 and 600 nodes 65

3. Portion of good active nodes in the total number of active nodes for different sparseness for graphs of 600 nodes 66

4. Contribution of different transformations to factor of \( m \) in \textit{decrease\_key} operation for graphs of 600 nodes 68

5. Factor for \( n^2 \) in Dijkstra's algorithm without heap for complete graphs 70

6. Factor for \( n^2 \) in Dijkstra's algorithm w/o heap for different sparseness for graphs of 600 nodes 71

7. Cost of Dijkstra's algorithm with and without R²-heaps for different sparseness 73

8. Number of CCTs and its ratio to \( n^2 \) 122
## FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>The recursive definition of binomial tree $B_k$</td>
<td>8</td>
</tr>
<tr>
<td>1-2</td>
<td>Binomial heap of 11 nodes</td>
<td>9</td>
</tr>
<tr>
<td>2-1</td>
<td>Pair transformation</td>
<td>14</td>
</tr>
<tr>
<td>2-2</td>
<td>Cleaning transformation</td>
<td>16</td>
</tr>
<tr>
<td>2-3</td>
<td>Sibling transformation</td>
<td>17</td>
</tr>
<tr>
<td>2-4</td>
<td>Run transformation I</td>
<td>23</td>
</tr>
<tr>
<td>2-5</td>
<td>Run transformation II</td>
<td>24</td>
</tr>
<tr>
<td>3-1</td>
<td>Architecture of $R^2$-heaps</td>
<td>29</td>
</tr>
<tr>
<td>3-2</td>
<td>Pair transformation in the child-grandchild situation</td>
<td>38</td>
</tr>
<tr>
<td>3-3</td>
<td>Sibling transformation in the $R^2$-heap</td>
<td>42</td>
</tr>
<tr>
<td>5-1</td>
<td>Portion of good active nodes for different sparseness</td>
<td>66</td>
</tr>
<tr>
<td>5-2</td>
<td>Comparison of Dijkstra's algorithm with and w/o heaps for different sparseness</td>
<td>74</td>
</tr>
<tr>
<td>7-1</td>
<td>Path Length Order Preservation</td>
<td>84</td>
</tr>
<tr>
<td>7-2</td>
<td>Canonical transformation for graph of 5 nodes</td>
<td>86</td>
</tr>
<tr>
<td>8-1</td>
<td>Mnemonic representation of the system of double linear inequalities</td>
<td>92</td>
</tr>
<tr>
<td>8-2</td>
<td>Algebraic representation for a graph of four nodes</td>
<td>94</td>
</tr>
<tr>
<td>8-3</td>
<td>Solution of the system of double linear inequalities by the modified Fourier method</td>
<td>98</td>
</tr>
<tr>
<td>9-1</td>
<td>Geometric representation of solution of the system of double linear inequalities for a graph of size 4</td>
<td>100</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>9-2</td>
<td>Primitive Shortest Paths</td>
<td>101</td>
</tr>
<tr>
<td>10-1</td>
<td>Successive CCT transformations</td>
<td>105</td>
</tr>
<tr>
<td>10-2</td>
<td>Interconnection between different representations of the problem</td>
<td>106</td>
</tr>
<tr>
<td>10-3</td>
<td>Zero - SPT</td>
<td>108</td>
</tr>
<tr>
<td>10-4</td>
<td></td>
<td>112</td>
</tr>
<tr>
<td>10-5</td>
<td></td>
<td>114</td>
</tr>
<tr>
<td>10-6</td>
<td></td>
<td>115</td>
</tr>
<tr>
<td>10-7</td>
<td></td>
<td>116</td>
</tr>
<tr>
<td>12-1</td>
<td>SPT for a graph of 100 nodes</td>
<td>128</td>
</tr>
</tbody>
</table>
I. R²-HEAPS WITH SUSPENDED RELAXATION FOR MANIPULATING 
PRIORITY QUEUES

CHAPTER 1

INTRODUCTION

1.1 Review of previous work

In the last twenty years, there has been an explosive growth in the field of combinatorial algorithms. Good algorithm design has been going hand-in-hand with good data structure design, and efficient interplay of data structure, combinatorics, and graph theory has resulted in the most effective algorithms.

Most of combinatorial algorithms manipulate dynamic sets, the sets changing over time, and support a variety of operations on them, like insert element into, delete element from, extracting the smallest element from, and others. The best way to implement dynamic sets depends upon the efficiency of the operations that must be supported.

In the family of data structures, the priority queues, or heaps, are recognized as a useful abstraction due to the large number of applications in which set
manipulations operations should be implemented. A variety of applications, which directly require using priority queues, include such fundamental problems in graph algorithms as finding shortest paths, minimal cost spanning tree, and other network optimization problems.

The priority queue structures include such obvious structures as the Sorted Linear List, and such sophisticated data structures as Fibonacci Heaps. The term "heap" was introduced by J. Williams in 1964 in connection with his heapsort algorithm. His heap structure uses linkless representation of a complete binary tree, storing the root in location 1, and the offspring of the node in location \( k \), in locations \( 2k \) and \( 2k+1 \). A heap is further characterized by the requirement that the key contained in any node must be no larger than the keys of its offspring; a tree with this property is said to be heap-ordered. Over the years, the term "heap" has acquired more general connotation, and is quite often used as an equivalent to the term priority queue. We will follow that convention.

Vuillemin [V78] invented a class of heaps, called binomial queues that support all heap operations in \( O(\log n) \) worst case time, here \( n \) is the number of items in the heap. Binomial heaps demonstrate improved complexity for

---

1 All logarithms here and further are to the base 2.
union operation while keeping already known time bounds for most operations supported by binary heaps [CLR92]. We will give a detailed description of the binomial queues in section 1.2.

Brown [B78] studied alternative representations of binomial heaps and developed experimental running time bounds. He showed the binomial queue is not just another data structure, but is the most practical structure for priority queues in many situations.

Fredman and Tarjan [FT87] developed an extension of binomial queues called Fibonacci heaps, abbreviated as F-heaps, that support delete-min and delete in \(O(\log n)\) amortized time and all other heap operations, in particular decrease-key in \(O(1)\) amortized time. For situations in which the number of deletions is small compared to the total number of operations, F-heaps are asymptotically faster than binomial queues. Thus F-heaps allow to obtain asymptotically faster algorithms for several well-known network optimization problems where the number of deletions is relatively small. For instance, it speeds up Dijkstra's algorithm for single-source shortest path problem with nonnegative length edges to \(O(m + n \log n)\) time, where \(n\) is the number of vertices and \(m\) is the number of edges in the graph. Using Dijkstra's algorithm as a subroutine results in a corresponding improvement of various other network algorithms.
Fibonacci heaps differ from binomial heaps in that they have a more relaxed structure, allowing work that maintains structure to be delayed until it is convenient to perform. That results in improved asymptotic time bound, however, the running time for Fibonacci heaps is amortized, not worst-case, time. Amortized analysis is used here to show that in the worst case, an average cost of an operation is small, even though a single operation might be expensive. (Probability is not involved in the amortized analysis.)

Peterson [P87] proposed Vheaps, a data structure that works with binary trees and uses a variant heap order than the usual one. It allows rotations within trees in order to balance them after decrease_key operation that causes a cut of the updated node. This technique has some similarity with the one used in F-heaps, but they are not identical in that while F-heaps produce cascading cuts, Vheaps produce cascading rotations. It was shown that Vheaps are just as asymptotically efficient as F-heaps, giving the same amortized time bounds.

In "Introduction to Algorithms" by T. Cormen, C. Leiserson, and R. Rivest [CLR92], the authors state: "From a practical point of view, however, the constant factors and programming complexity of Fibonacci heaps make them less desirable than ordinary binary...heaps for most applications. Thus, Fibonacci heaps are predominantly of theoretical interest. If a much simpler
data structure with the same amortized time bounds as Fibonacci heaps were developed, it would be of great practical use as well."

Driscoll, Gabow, Shrairman, and Tarjan [DGST88] presented another extension of binomial queues, a new data structure called Relaxed Heaps which does answer this appeal: it presents an alternative to Fibonacci heaps, a much simpler data structure which demonstrates theoretical and practical improvements over the Fibonacci heaps.

The Relaxed heaps allow more relaxation than F-heaps: it is allowed not only to delay work to maintain structure, but also to keep a limited amount of violations in the heap structure. Furthermore, while the Fibonacci heaps use the relaxation of the structural organization itself, the Relaxed heaps use the relaxation in the key relations only, keeping the structure unchanged. The strict structural discipline of the Relaxed heaps results in the improved complexity for the worst case vs. amortized time bound, namely $O(1)$ for decrease_key and $O(\log n)$ for delete_min over the Fibonacci heaps in practical implementation as well.

Two versions of the Relaxed heaps are given. The first, Rank Relaxed Heaps achieves the same amortized bounds as Fibonacci heaps, but maintains more structure. The second, Run Relaxed Heaps gives theoretical improvement
over *Fibonacci heaps*: it achieves the above time bound for *decrease_key* and *delete_min* in the *worst case*, rather than in the amortized case.

*Relaxed heaps* give also a processor-efficient parallel implementation of the Dijkstra’s shortest path algorithm, and hence other algorithms for network optimization [DGST88].

This research presents a further development of *Relaxed heaps: Heaps with Suspended Relaxation*, abbreviated as *R2-heaps*. *R2-heaps* combine the strength of both *Rank Relaxed heaps* and *Run Relaxed heaps*: while it keeps the simplicity of the first, it achieves the same *worst case* time bound for *decrease_key* and *delete_min* operations as the second does. That contributes to the name abbreviation *R2-heaps* (*"R square heaps"*).

*R2-heaps* use some new concepts and disciplines in structural organization and manipulation of relaxed heaps that allowed to cut the number of main transformations from six to three, and to decrease the number of pointers per node from 4 to 1 compared to the Relaxed heaps [DGST88]. The experiment run on more than 400 randomly generated graphs in the range of sizes 20 through 600 nodes with different sparseness proved that the *R2-heaps* can be implemented robustly and efficiently. The experiment demonstrated quite
low factors for $m$ and $n \log n$ fractions in the worst case time bound for
Dijkstra’s algorithm$^2$: $3m + 22n \log n$.

1.2 Definitions and notations

A priority queue, or heap, is a data structure for storing and manipulating a set
of items $x$, each having a numerical key denoted $k(x)$.

The main operations are:

- `make_heap` initialize a heap to store the empty set;
- `insert(x)` make $x$ a new item in the heap;
- `decrease_key` decrease key $k(x)$ to a smaller value $v$;
- `delete_min` delete an item of min key from the heap and return it as $x$;
- `find_min` return an item of minimum key as $x$;
- `delete(x)` delete item $x$ from the heap.

A binomial heap, or queue, is a collection of heap-ordered binomial trees.
The binomial trees $B_r$ are defined recursively as follows:

---

$^2$ The factors were determined based on operations count. The formula uses the rounded numbers. The detailed experimental data is presented in Chapter 5.
$B_0$ is one node; $B_k$ consists of two $B_{k-1}$ trees, the root of one being defined as a child of the root of the other, see Fig. 1-1. The integer $k$ is called the index, or rank of such a binomial tree.

![Fig. 1-1 The recursive definition of binomial tree $B_k$](image)

In all figures, a triangle labeled $r$ represents the binomial tree $B_r$. For any $k$, $0 \leq k \leq r$, $B_{r+1}$ consists of a $B_k$ tree with additional children of the root that are themselves roots of $B_{k}, B_{k+1}, ..., B_r$ trees.

Rank of an arbitrary tree in a binomial queue is the longest path from a leaf of this tree to its root. Rank of $B_0$ is 0. For any node $x$ in a binomial tree, $\text{rank}(x)$ is the rank $r$ of the maximal subtree $B_r$ rooted at $x$. A binomial tree is an ordered tree, with the children of a node ordered by increasing rank. The last child of a node is the child of highest rank. The binomial tree $B_r$ has $2^r$ nodes and height $r$. A binomial queue for $2^r$ items is a heap-ordered tree $B_r$. A binomial queue for $n$ items, $n$ arbitrary, consists of at most $\lfloor \log n \rfloor + 1$ heap-ordered binomial trees, a tree corresponding each to "one" bit in the binary
expansion of $n$. An example of the Binomial heap of 11 nodes is presented on Fig. 1-2.

\[
\begin{array}{cccc}
2^3 & + & 0 & + 2^1 & + 2^0 & = & 11 \\
1 & 0 & 1 & 1
\end{array}
\]

It is clear from the definition, that all binomial trees having a given index are isomorphic in the sense that they have the same shape, and all binomial heaps are isomorphic given the total number of nodes.
Chapter 2

Relaxed Heaps

2.1 Definitions

A relaxed heap, abbreviated as R-heap, is a type of binomial queue that allows heap order to be partially violated.

In the tree data structure, each node stores one item, including its key. In our examples, a tree node represents a node of the graph with its label equal to a node key in the tree.

In a heap-ordered tree, the parent's key is less or equal to the key of its child:

\[ k(p) \leq k(c) \]

If node \( c \) is the root, or satisfies this relation we call it good, otherwise, if \( k(p) > k(c) \), it is bad. In a heap-ordered structure, all nodes are good. In a Relaxed heap, we allow a limited number of violations in the key relations.
If a `decrease_key` operation is applied to a node, this node is considered as *active*, since it represents a potential violation of the structure, and it could be *good* or *bad*. Active node can cease being bad by changing its parent or as a result of its parent key change. The actual status of an active node is checked at the beginning of the appropriate transformation, and if the node is *good*, it is deactivated.

Let $\alpha$ denote the number of *active* nodes in the entire collection of trees at any point of the algorithm. Initially there are no active nodes, so $\alpha$ is zero. The `decrease_key` resets $k(x)$ to $v$, and each `decrease_key` increases $\alpha$ by one. For each transformation, assume `decrease_key` has created an active node $a$ of rank $r$ with parent $p$ and grandparent $g$.

One of the main ideas of the Relaxed Heaps is to make *bad* children, that violate the heap order, "play together" in such a way that at least one *bad* child becomes *good*. A variety of group transformations are applied in order to organize this play, and as a result to decrease the number of bad children in $O(1)$ time.

In two implementations of R-heaps, the *Rank Relaxed Heaps* (*Rank R-heaps*) allow just one *active node per rank*, while the *Run Relaxed Heaps* (*Run R-heaps*) are less stringent and allow *runs* of *bad* nodes.
We will focus our attention mostly on implementation of \textit{decrease_key} and \textit{delete_min} operations since their efficiency is crucial for improvement of Dijkstra's algorithm asymptotic time bound, and that was our goal and the main motivation in designing the \textit{Relaxed Heaps}. It will be shown that the \textit{Relaxed Heap} transformations allow us to achieve $O(m + n \log n)$ \textit{amortized} time with use of the \textit{Rank R-heaps} and the same, but \textit{worst case} time bound with use of the \textit{Run R-heaps}.

Now we will consider each of these implementations in greater details.

\subsection*{2.2 Rank Relaxed Heaps}

A \textit{Rank R-heap} is a relaxed binomial queue that satisfies two conditions:

a) For any $r$, there is at most one active node of rank $r$;

A variety of transformations - \textit{pair transformation}, \textit{good sibling} and \textit{active sibling} transformations, as well as \textit{cleaning} are used in order to support the legal \textit{Rank Relaxed Heap} structure.

We claim that each transformation takes $O(1)$ time and either

\footnote{$b$ is not crucial}
i. decrease $\alpha$, or

ii. does not change $\alpha$ and does not require further transformations to be executed.

For a Rank $R$-heap, if a second active node appears at a given rank, we apply one or a combination of the transformations in order to correct the structure.

We will present several transformations, those that help to illustrate the main ideas used in the Relaxed heaps and those that continue to be used in the next generation of Relaxed Heaps, called $R^2$-heaps with suspended relaxation.

The following notations are used in pictorial illustrations.

An edge joining a child $c$ to its parent $p$ is labeled in four different ways: an arrow from $c$ to $p$ indicates that $c$ is good; cross mark on the edge indicates $c$ is active and bad, dotted edge indicates $c$ may be a new active node; no mark means the status of $c$ is unknown. Each illustration represents a fragment of the heap.

2.2.2 Transformations