Some Notes on the Theory of Hilbert Spaces of Analytic Functions of the Unit Disc

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To

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Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. This work deals with some aspects of the theory of Hilbert spaces of analytic functions on $\mathbb{D}$.

H^2 is defined to be the set of holomorphic functions on $\mathbb{D}$ with square summable Maclaurin coefficients. The norm of an element f of H^2 can be given by a series, an integral over $\partial\mathbb{D}$ or an area integral over $\mathbb{D}$:

$$\|f\|^2_{\mathrm{H}^2} = \sum_{n\geq 0} |\hat{f}(n)|^2 \tag{0.1}$$

$$\|f\|^2_{\mathrm{H}^2} = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \tag{0.2}$$

$$\|f\|^2_{\mathrm{H}^2} = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \tag{0.3}$$

where $d\theta$ is the Lebesgue measure on $\partial\mathbb{D}$, and dA is the normalized Lebesgue measure on $\mathbb{D}$, $dA(re^{i\theta}) = (1/\pi) r d\theta dr$. The values on $\partial\mathbb{D}$ are the radial limits:

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

See [5, 6].

An important subset of H^2 is the Dirichlet space, D, of the holomorphic functions on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\,; \tag{0.4}$$

the norm of f is given by

$$\|f\|^2_{\mathrm{D}} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)\,\cdot \tag{0.5}$$

In [9] Richter introduced a family of spaces between the Dirichlet space D and Hardy space H^2 which we briefly describe now. Let μ be a finite positive Borel measure on $\partial\mathbb{D}$. Define the harmonic function ϕ_μ on $\mathbb{D}$ by

$$\phi_\mu(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta)\cdot \tag{0.6}$$

If $\mu = 0$ define $\mathrm{D}(\mu) = \mathrm{H}^2$; otherwise $\mathrm{D}(\mu)$ is defined to be the space of analytic functions on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f'(z)|^2 \phi_\mu(z) dA(z) < \infty\,; \tag{0.7}$$

$\mathrm{D}(\mu)$ is a Hilbert space with norm given by

$$\|f\|^2_{\mathrm{D}(\mu)} = \|f\|^2_{\mathrm{H}^2} + \int_{\mathbb{D}} |f'(z)|^2 \phi_\mu(z) dA(z)\cdot \tag{0.8}$$

We now describe another technique to produce subspaces of H^2; this one is a particular case of a construction in [3], see also [10]. Let $A : \mathrm{H}^2 \to \mathrm{H}^2$ be a bounded linear operator. As a set $M(A)$ is the range of A. The norm is defined by

$$\|A(x)\|_{M(A)} = \|x\| \text{ for all } x \perp ker(A) \tag{0.9}$$

In [9] it is shown that $\mathrm{D}(\delta_\zeta) = \mathrm{M}(\overline{\zeta} - \mathrm{S}^*)$ where S is the unilateral shift on H^2. The equivalence of the norms carried by those spaces will permit us to define an operator $A \in L(\mathrm{H}^2)$ satisfying

$$\|A(f)\|_{\mathrm{H}^2} = \|f\|_{\mathrm{D}(\delta_1)}$$

which we'll study in some detail. It will be shown that it is a rank one perturbation of a Toeplitz operator with real valued symbol.

These results will be generalized to operators of the form

$\prod_{i=1}^n (\overline{\zeta_i} - S^*)$ where $\zeta_i \in \partial\mathbb{D}$ for $i = 1, \cdots, n$. We'll show that

$$M\left(\prod_{i=1}^n (\overline{\zeta_i} - S^*)\right) = \mathrm{D}\left(\sum_1^{\mathrm{n}} \delta_{\zeta_{\mathrm{i}}}\right)$$

and that the associated operator A is a rank n perturbation of a Toeplitz operator. We will perform a similar analysis for for the space $M\left((1 - S^*)^2\right)$.

Douglas, in [4], used a kind of integral Richter in [8] called a local Dirichlet integral. For a function f and $\zeta \in \partial\mathbb{D}$, it is denoted by $D_\zeta(f)$, and has the following definition:

$$D_\zeta(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{i\theta}) - f(\zeta)}{e^{i\theta} - \zeta} \right|^2 d\theta\cdot \tag{0.10}$$