Some Notes on the Theory of Hilbert Spaces of Analytic Functions of the Unit Disc

by

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ISBN: 1-58112-023-0

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1998
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M. A. (University of California at Berkeley) 1991

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics in the GRADUATE DIVISION of the UNIVERSITY of CALIFORNIA at BERKELEY

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Jorge-Nuno O. Silva
Several people helped me in the process of coming to Berkeley and living here. Among them Tone Feijó, Carlos Sarrico and Jonathan Walden. I thank them very much.

I thank JNICT-Programa CIÊNCIA and Fundação Luso-Americana para o Desenvolvimento for their financial support.

I am grateful to Professor D. Sarason for his help and advice.

To José Luís Fachada, João Santos Guerreiro and Nuno Costa Pereira, after all these years, my thanks.

I thank my family – Laura, Manuel and Raquel – for their love and support.
To
Laura and Manuel.
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Introduction

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. This work deals with some aspects of the theory of Hilbert spaces of analytic functions on $\mathbb{D}$.

$H^2$ is defined to be the set of holomorphic functions on $\mathbb{D}$ with square summable Maclaurin coefficients. The norm of an element $f$ of $H^2$ can be given by a series, an integral over $\partial\mathbb{D}$ or an area integral over $\mathbb{D}$:

\[
\|f\|_{H^2} = \sum_{n \geq 0} |\hat{f}(n)|^2
\]

\[
\|f\|_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta
\]

\[
\|f\|_{H^2} = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z)
\]

where $d\theta$ is the Lebesgue measure on $\partial\mathbb{D}$, and $dA$ is the normalized Lebesgue measure on $\mathbb{D}$, $dA(re^{i\theta}) = (1/\pi)r d\theta dr$. The values on $\partial\mathbb{D}$ are the radial limits:

\[
f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})
\]

See [5, 6].

An important subset of $H^2$ is the Dirichlet space, $D$, of the holomorphic functions on $\mathbb{D}$ with

\[
\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty
\]

the norm of $f$ is given by

\[
\|f\|_{D}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z)
\]

In [9] Richter introduced a family of spaces between the Dirichlet space $D$ and Hardy space $H^2$ which we briefly describe now. Let $\mu$ be a finite positive Borel measure on $\partial\mathbb{D}$. Define the harmonic function $\phi_\mu$ on $\mathbb{D}$ by
\[ \phi_{\mu}(z) = \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta). \]  

(0.6)

If \( \mu = 0 \) define \( D(\mu) = H^2 \); otherwise \( D(\mu) \) is defined to be the space of analytic functions on \( D \) with

\[ \int_D |f'(z)|^2 \phi_{\mu}(z) dA(z) < \infty; \]  

(0.7)

\( D(\mu) \) is a Hilbert space with norm given by

\[ \|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + \int_D |f'(z)|^2 \phi_{\mu}(z) dA(z). \]  

(0.8)

We now describe another technique to produce subspaces of \( H^2 \); this one is a particular case of a construction in [3], see also [10]. Let \( A : H^2 \to H^2 \) be a bounded linear operator. As a set \( M(A) \) is the range of \( A \). The norm is defined by

\[ \|A(x)\|_{M(A)} = \|x\| \text{ for all } x \perp ker(A) \]  

(0.9)

In [9] it is shown that \( D(\delta_{\zeta}) = M(\zeta - S^*) \) where \( S \) is the unilateral shift on \( H^2 \). The equivalence of the norms carried by those spaces will permit us to define an operator \( A \in L(H^2) \) satisfying

\[ \|A(f)\|_{H^2} = \|f\|_{D(\delta_{\zeta})} \]

which we’ll study in some detail. It will be shown that it is a rank one perturbation of a Toeplitz operator with real valued symbol.

These results will be generalized to operators of the form

\[ \prod_{i=1}^n (\zeta_i - S^*) \]  

where \( \zeta_i \in \partial D \) for \( i = 1, \cdots, n \). We’ll show that

\[ M\left( \prod_{i=1}^n (\zeta_i - S^*) \right) = D\left( \sum_{i=1}^n \delta_{\zeta_i} \right) \]

and that the associated operator \( A \) is a rank \( n \) perturbation of a Toeplitz operator. We will perform a similar analysis for the space \( M((1 - S^*)^2) \).

Douglas, in [4], used a kind of integral Richter in [8] called a local Dirichlet integral. For a function \( f \) and \( \zeta \in \partial D \), it is denoted by \( D_{\zeta} (f) \), and has the following definition:

\[ D_{\zeta}(f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(e^{i\theta}) - f(e^{i\theta} - \zeta)}{e^{i\theta} - \zeta} \right|^2 d\theta. \]  

(0.10)