The Foundations of Real Analysis

A Fundamental Course with 347 Exercises and Detailed Solutions

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Contents

Chapter 1

The Set of Real Numbers

1.1 Introduction

Real Analysis must begin with the set of real numbers $\mathbb{R}$, along the the two binary operations addition $+$ and multiplication $\cdot$, which makes $(\mathbb{R},+,\cdot)$ a field in the algebraic sense. In fact $(\mathbb{R},+,\cdot)$ can be called the only *complete, ordered field*, with "only" being up to algebraic isomorphism of rings.

It is the completeness property of $\mathbb{R}$ that is essential to calculus, and many of its results. Note that $(\mathbb{Q},+,\cdot)$ is an ordered field, where $\mathbb{Q}$ being the set of rational numbers $\{\frac{m}{n} : m, n \text{ integers, and } n \neq 0\}$. However $\mathbb{Q}$ is not complete. To see this, take for instance terms in the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, \cdots$$

that is the sequence of terms $\{a_n\}_{n=0}^{\infty}$ which have the decimal expansion for $\sqrt{2}$ out to the $(10)^{-n}th$ place. Clearly all the terms in this sequence are rational, however, if the sequence converges, it must converge to $\sqrt{2} \notin \mathbb{Q}$. Thus $\mathbb{Q}$ is not complete.

1.2 Fields and the Set of Rational Numbers

1.2.1 Fields in the Algebraic Sense

Let $\mathbb{F}$ be a set that has the following binary relations $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, called addition and multiplication respectively. We will assume

that the triple $(\mathbb{F},+,\cdot)$ satisfies the following list of properties:

For any $x, y, z \in \mathbb{F}$

1. **Commutativity of Addition:**

$$x + y = y + x$$

2. **Associativity of Addition:**

$$x + (y + z) = (x + y) + z$$

3. **Existence of Zero:** There is an element $0 \in \mathbb{F}$, called **zero**, which satisfies

$$0 + x = x$$

for all $x \in \mathbb{F}$.

4. **Additive Inverse:** For each $x \in \mathbb{F}$ there is a $-x \in \mathbb{F}$ so that

$$x + (-x) = 0.$$

5. **Commutativity of Multiplication:**

$$x \cdot y = y \cdot x$$

6. **Associativity of Multiplication:**

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

7. **Existence of Unity**[1]**:** There is an element $1 \in \mathbb{F}$, called **one**, so that

$$1 \cdot x = x$$

for all $x \in \mathbb{F}$.

8. **Multiplicative Inverse:** For each non-zero $x \in \mathbb{F}$ there is a $x^{-1} \in \mathbb{F}$ so that

$$x \cdot x^{-1} = 1$$

[1] Unity in a ring refers to 1.

9. **Distributive Property:** $+$ and $\cdot$ are related by

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

Needless to say $1, 0 \in \mathbb{F}$ may not be the $1, 0$ in the set of real numbers.

Such an ordered triple $(\mathbb{F}, +, \cdot)$ is called a **field**.

The Set of Natural Numbers

There is an important set of numbers called the **natural numbers**[2] , denoted by $\mathbb{N}$, and which satisfies the property that $1 \in \mathbb{N}$, and if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. Moreover, there does not exists an $m \in \mathbb{N}$ so that $m + 1 = 1$. Here of course $\mathbb{N}$ has two binary operations $+, \cdot$.[3] Moreover, the set of natural numbers is closed under $+$ and $\cdot$ and satisfies the following list of properties for any $k, m, n \in \mathbb{N}$:

1. **Associativity:**

$$k + (m + n) = (k + m) + n, \quad k \cdot (m \cdot n) = (k \cdot m) \cdot n$$

2. **Commutativity:**

$$k + m = m + k, \quad k \cdot m = m \cdot k$$

3. **Existence of a Unity:** There is an element $1 \in \mathbb{N}$, called **one**, so that

$$1 \cdot k = k$$

for any $k \in \mathbb{N}$.

[2]The natural numbers $\mathbb{N}$ can actually be characterized by the following five axioms: A1: $1 \in \mathbb{N}$. A2: $\forall x \in \mathbb{N}$ there exists a unique $(x + 1) \in \mathbb{N}$, called the successor of x. A3: $\forall x \in \mathbb{N}$, $x + 1 \neq 1$. That is 1 is not the successor of any natural number. A4: If $x + 1 = y + 1$, then $x = y$. A5 (the axiom of induction): Suppose $A \subseteq \mathbb{N}$ satisfies: $1 \in A$; $x \in A$ implies $(x + 1) \in A$. Then $A = \mathbb{N}$. For a complete construction of the properties of $\mathbb{N}$ see for instance [7].

[3]Although we are using the same notation for $+, \cdot$ for both our field $\mathbb{F}$ and $\mathbb{N}$, as well as for the element $1 \in \mathbb{F}$ and $1 \in \mathbb{N}$, they unrelated in general.

4. **Cancellation Property:**

$$k + m = k + n \Rightarrow m = n$$

$$k \cdot m = k \cdot n \Rightarrow m = n$$

5. **The Distributive Property:**

$$k \cdot (m + n) = (k \cdot m) + (k \cdot n)$$

6. For a given pair of natural numbers m, n one and only one of the following holds:

   - $m = n$,
   - $m = n + k$ for some $k \in \mathbb{N}$,
   - $n = m + k$ for some $k \in \mathbb{N}$.

   This in turn allows us to define an ordering on $\mathbb{N}$. The following are **order properties:**

   (a) $m < n$ if and only if there exists $j \in \mathbb{N}$ so that $n = m + j$.

   (b) $m \leq n$ if and only if $m < n$ or $m = n$.

   (c) $k < m$ and $m < n$ implies $k < n$.

   (d) $m \leq n$ and $n \leq m$ implies $m = n$

   (e) $m < n$ implies $m + k < n + k$ and $k \cdot m < k \cdot n$.

Moreover, we also have the following axiom for the set $\mathbb{N}$ of natural numbers:

The Axiom of Induction: Let $S \subseteq \mathbb{N}$. Suppose S satisfies

1. $1 \in S$.

2. $n \in S$ implies $n + 1 \in S$.

Then $S = \mathbb{N}$.

From this characterization of $\mathbb{N}$ in terms of the above properties, we see that $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$.

The Set of Integers

From the set of natural numbers we may define the set of **integers**, denoted by[4] $\mathbb{Z}$, that has the same binary operations $+, \cdot$ and is closed under these operations.[5] $\mathbb{Z}$ contains $\mathbb{N}$ as a subset, and also contains an element 0 so that $m + 0 = m$ for all $m \in \mathbb{Z}$. Moreover, for any $m \in \mathbb{Z}$ we have $m = 0$, $m \in \mathbb{N}$ or $-m \in \mathbb{N}$, where $m + (-m) = 0$. Thus we see that $\mathbb{Z} = \{\cdots, -3, -2, -2, 0, 1, 2, 3, 4, \cdots\}$. $\mathbb{Z}$ satisfies all the field axioms except the axiom which says that every non-zero element x has a multiplicative inverse x^{-1}. Moreover, the set of integers also satisfies the **cancellation property**[6], and so $\mathbb{Z}$ cannot have any zero divisors, that is

$$x \cdot y = 0 \quad \Rightarrow \quad x = 0, \quad \text{or} \quad y = 0.$$

Moreover, on $\mathbb{Z}$ we may define an ordering via the relation $<$, where $m < n$ if and only if $n + (-m) \in \mathbb{N}$.

The Set of Rational Numbers

From the set of integers, we may construct set set of **rational numbers**, denoted by $\mathbb{Q}$, which are defined by

$$\mathbb{Q} := \{a \cdot b^{-1} : a, b \in \mathbb{Z}, b \neq 0\}. \tag{1.1}$$

We may think of this as the algebraic extension of the integral domain $\mathbb{Z}$ to a field of fractions. In a later section we shall give a rigorous construction of $\mathbb{Q}$ from $\mathbb{Z}$, as well as a rigorous construction of $\mathbb{Z}$ from $\mathbb{N}$.

Notational Conventions for Fields

The following is a list of standard notational conventions:

1. We often write xy instead of $x \cdot y$.
2. We define **subtraction** by

$$x - y := x + (-y).$$

[4] $\mathbb{Z}$ for the German *Zahl*, which means number.
[5] In a later section we shall give a rigorous argument for the construction of $\mathbb{Z}$ from $\mathbb{N}$.
[6] $a \cdot b = a \cdot c$ and $a \neq 0$ implies $b = c$

3. We often write $\frac{1}{x}$ instead of x^{-1}.

4. We define **division** by

$$x \div y \text{ or } \frac{x}{y} := x \cdot y^{-1}$$

for $x \in \mathbb{F}$, and $y \neq 0$ in $\mathbb{F}$.

5. We define **powers** of a element x by $x^1 := x$, $x^2 := x \cdot x$, $x^3 := x \cdot x^2 = x \cdot x \cdot x$ and for n a positive integer

$$x^{n+1} := x \cdot x^n.$$

For n a positive integer, we define x^{-n} as $(x^{-1})^n$ or equivalently $(x^n)^{-1}$. Since both ways to define this will agree with one another.

6. We define **multiples** of an element x by $1x := x$, $2x := x + x$, $3x := x + 2x = x + x + x$, and for a positive integer n by

$$(n+1)x = x + nx.$$

Moreover, because of the distributive property, we have

$$n \cdot x = nx.$$

That is,

$$\underbrace{(1 + 1 + \cdots + 1)}_{n \text{ times}} \cdot x = \underbrace{x + x + \cdots + x}_{n \text{ times}}.$$

Note as well

$$\underbrace{(1 + 1 + \cdots + 1)}_{n \text{ times}} \cdot x$$

where here $1 \in \mathbb{F}$.

7. **Order of Operations:**

   (a) Operations in parentheses are performed first, working with the inner most first. In the innermost parentheses, or in the absence of parentheses, follow the remaining steps.

   (b) Powers are computed first, going from left to right.

(c) Multiplication and division are performed next (with equal weight) going from left to right.

(d) Addition and subtraction are performed after all multiplications and divisions have been performed. Addition and subtraction (which hold equal weight) are performed from left to right.

Thus, we see that we may write the distributive property as

$$x(y+z) = xy + xz.$$

Some Consequences of the Field Axioms

The following result shows that additive and multiplicative inverses for a fixed element x are unique, and the elements 1 and 0 are unique in a field $\mathbb{F}$.

Proposition 1.2.1 *The elements* 0 *and* 1 *of a field are unique. Moreover, for a given element $x \in \mathbb{F}$, the additive inverse $-x$ is unique, and for $x \neq 0$ the multiplicative inverse x^{-1} is unique.*

Proof: Suppose that there are two elements $0_1, 0_2 \in \mathbb{F}$ such that $0_1 + x = x$ and $0_2 + x = x$ for any $x \in \mathbb{F}$. Then $0_1 + 0_2 = 0_2$ by using $x = 0_2$, and $0_1 + 0_2 = 0_1$ by using $x = 0_1$. Thus, $0_1 = 0_2$. Next, we suppose there are two elements $1_1, 1_2 \in \mathbb{F}$ such that $1_1 \cdot x = x$ and $1_2 \cdot x = x$. Then $1_1 \cdot 1_2 = 1_2$ by using $x = 1_2$ and on the other hand $1_1 \cdot 1_2 = 1_1$ by using $x = 1_1$. Thus, $1_1 = 1_2$.

Now, given $x \in \mathbb{F}$, suppose there are two elements $y, z \in \mathbb{F}$ so that $x + y = 0$ and $x + z = 0$. Then, by associativity we have

$$y = y + 0 = y + (x + z) = (y + x) + z = 0 + z = z.$$

Given $x \neq 0$, suppose there are two elements $y, z \in \mathbb{F}$ so that $xy = 1$ and $xz = 1$. Then, by associativity we have

$$y = y \cdot 1 = y(xz) = (yx)z = 1 \cdot z = z.$$

□

Since additive and multiplicative inverses are unique,

$$-(-x) = x, \quad \frac{1}{\frac{1}{x}} = (x^{-1})^{-1} = x \tag{1.2}$$

Another important result to note is the **cancellation property:**

Proposition 1.2.2 *If $x + y = x + z$, then $y = z$. Moreover, if $x \neq 0$ and $xy = xz$, then $y = z$.*

Proof: If $x + y = x + z$, by adding $-x$ to both sides we get

$$y = 0 + y = (-x + x) + y = -x + (x + y)$$

$$= -x + (x + z) = (-x + x) + z = 0 + z = z.$$

Moreover, if $x \neq 0$ and $xy = xz$, by multiplying both sides by x^{-1} we get

$$y = 1 \cdot y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}(xz) = (x^{-1}x)z = 1 \cdot z = z.$$

□

The next result relates $(-1) \cdot x$ to $-x$ and also gives us the property $0 \cdot x = 0$, neither of which are axioms of our field $\mathbb{F}$.

Proposition 1.2.3 *For any $x \in \mathbb{F}$ we have $(-1) \cdot x = -x$ and $0 \cdot x = 0$. Moreover, $(-1)^2 = 1$.*

Proof: $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$ and thus by the cancellation property we have

$$0 \cdot x = 0.$$

To show $(-1) \cdot x = -x$ we will show $(-1) \cdot x$ is the additive inverse $-x$, and thus by uniqueness of inverses, we get the result. To see this, note

$$(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1 + 1) \cdot x = 0 \cdot x = 0.$$

For the last result, we will show $(-1)^2$ is the additive inverse of -1.

$$-1 + (-1)^2 = (-1) \cdot 1 + (-1)^2 = (-1) \cdot (1 + (-1)) = (-1) \cdot 0 = 0.$$

□

A consequence of the above proposition is that for any $x, y \in \mathbb{F}$ we have

$$(-x) \cdot y = -(x \cdot y) = x \cdot (-y) \tag{1.3}$$

To see this, we use the property $-x = (-1) \cdot x$, and thus

$$(-x) \cdot y = ((-1) \cdot x) \cdot y = (-1) \cdot (x \cdot y) = -(x \cdot y).$$

To see the other equality, we use commutativity, and reverse the roles of x and y.

Moreover, we also have

$$(-x) \cdot (-y) = x \cdot y,$$

since

$$(-x) \cdot (-y) = (-1)^2 \cdot x \cdot y.$$

The next result tells us that there are no zero divisors in a field.

Proposition 1.2.4 *Suppose that $x \cdot y = 0$, then either $x = 0$ or $y = 0$.*

Proof: Suppose $x \cdot y = 0$ and $x \neq 0$. Then multiplying $x \cdot y = 0$ by $\frac{1}{x}$ we get

$$0 = \frac{1}{x} \cdot 0 = \frac{1}{x} \cdot (x \cdot y) = (\frac{1}{x} \cdot x) \cdot y = 1 \cdot y = y.$$

□

1.2.2 The Rational Numbers and the Integers, a Rigorous Discussion

In this section we will discuss the construction of the set of integers from the set of natural numbers, and then a construction of the set of rational numbers from the set of integers. We will think of the set of rational numbers as an algebraic extension of the set of integers (as an integral domain) to a field.

The Construction of the Integers from the Natural Numbers

Consider the set $\mathbb{N} \times \mathbb{N}$. On this set we define a relation[7] R by

$$(a, b) \sim_R (c, d) \quad \iff \quad a + d = b + c. \tag{1.4}$$

We claim that R is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. Clearly $(a, b) \sim_R (a, b)$ since $a + b = b + a$, and thus R is reflexive. Likewise, if $(a, b) \sim_R (c, d)$, then $a + d = b + c$. By commutativity of addition on $\mathbb{N}$, we know that $c + b = d + a$, and so $(c, d) \sim_R (a, b)$. Hence R is a symmetric relation on $\mathbb{N} \times \mathbb{N}$. Next, we suppose that $(a, b) \sim_R (c, d)$ and $(c, d) \sim_R (e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Adding f to the first equation $a + d = b + c$ we get $a + d + f = b + c + f$, and then using the second equation $c + f = d + e$ we get $b + c + f = b + d + e$, and so $a + d + f = b + d + e$. By the cancellation property of $\mathbb{N}$, we may conclude that $a + f = b + e$, and so $(a, b) \sim_R (e, f)$. Thus we see that R is a transitive relation on $\mathbb{N} \times \mathbb{N}$. This allows us to define $\mathcal{Z}$ to be the set of equivalence classes defined by this relation on $\mathbb{N} \times \mathbb{N}$. Let $[a, b]$ denote the equivalence class of $(a, b) \in \mathbb{N} \times \mathbb{N}$ with respect to this equivalence relation R.

On $\mathcal{Z}$ we define an operation $+$ by $[a, b] + [c, d] := [a + c, b + d]$. Likewise, we define an operation $\cdot$ by $[a, b] \cdot [c, d] := [ac + bd, ad + bc]$. Now it is our job to show that on $\mathcal{Z}$ the operations $+$ and $\cdot$ as defined are well-defined. Moreover, $\mathcal{Z}$ is closed with respect to $+$ and $\cdot$. On $\mathcal{Z}$, $+$ is commutative and associative, $\cdot$ is commutative and associative, and $+$ and $\cdot$ satisfy a distributive property.

First we address the well-defined nature of our definitions of $+$ and $\cdot$. Suppose $[a, b] = [a', b']$, and $[c, d] = [c', d']$. In other words, $(a, b) \sim_R (a', b')$ and $(c, d) \sim_R (c', d')$. We claim that $[a, b] + [c, d] = [a', b'] + [c', d']$ and $[a, b] \cdot [c, d] = [a', b'] \cdot [c', d']$. In order to see this, we first observe that for any $n \in \mathbb{N}$ we have that $[a, b] = [a + n, b + n]$, since $a + (b + n) = b + (a + n)$. Now,

$$\begin{aligned} [a, b] + [c, d] &= [a + c, b + d] \\ &= [a + c + a' + c', b + d + a' + c'] \\ &= [a + c + a' + c', a' + b + c' + d] \end{aligned}$$

[7] Our motivation is (a, b) should be thought of as $a - b$, since we may obtain $\mathbb{Z}$ from $\mathbb{N}$ by taking all possible differences of two natural numbers.

$$= [a + c + a' + c', a + c + b' + d']$$
$$= [a' + c', b' + d']$$
$$= [a', b'] + [c', d'].$$

Likewise,

$$[a, b] \cdot [c, d] = [ac + bd, ad + bc]$$
$$= [ac + bd + ac' + bd', ac' + ad + bd' + bc]$$
$$= [ac + bd + ac' + bd', a(c' + d) + b(d' + c)]$$
$$= [ac + bd + ac' + bd', a(c + d') + b(d + c')]$$
$$= [ac + bd + ac' + bd', ac + bd + c'b + d'a]$$
$$= [ac' + bd', c'b + d'a]$$
$$= [ac' + bd' + a'c' + b'd', c'a' + c'b + d'b' + d'a]$$
$$= [ac' + bd' + a'c' + b'd', c'(a' + b) + d'(b' + a)]$$
$$= [ac' + bd' + a'c' + b'd', c'(a + b') + d'(b + a')]$$
$$= [ac' + bd' + a'c' + b'd', ac' + bd' + a'd' + b'c']$$
$$= [a'c' + b'd', a'd' + b'c']$$
$$= [a', b'] \cdot [c', d'].$$

We leave it as an exercise to the reader to show the associativity and commutativity of both $+$ and $\cdot$ as defined on $\mathcal{Z}$.

Next, we address the distributive property, which relates the operations $+$ and $\cdot$.

$$[a, b] \cdot ([c, d] + [e, f]) = [a, b] \cdot [c + e, d + f]$$
$$[a(c + e) + b(d + f), a(d + f) + b(c + e)]$$
$$= [(ac + bd) + (ae + bf), (ad + bc) + (af + be)]$$
$$= [ac + bd, ad + bc] + [ae + bf, af + be]$$
$$= ([a, b] \cdot [c, d]) + ([a, b] \cdot [e, f]).$$

Next, we will show how to embed $\mathbb{N}$ into $\mathcal{Z}$, and we will do this via the identification of $\mathbb{N}$ with the elements of $\mathcal{Z}$ in the subset $\mathcal{P} := \{[m, n] : m > n\}$. First, we claim that $[m, n] \in \mathcal{P}$ has a representation of the form $[n+k, n]$.

This is clear from the fact that $m > n$ in $\mathbb{N}$ means that $m = n + k$ for some $k \in \mathbb{N}$. We claim that $\mathcal{P}$ is isomorphic to $\mathbb{N}$ relative to the operations $+, \cdot$ as defined on each set. Since we can think of elements of $\mathcal{P}$ as elements of the form $[n+k, n]$, we define the mapping $k \mapsto [n+k, n]$, where $n \in \mathbb{N}$ is any fixed natural number. Since $[k + m, m] = [k + n, n]$ for all $m, n \in \mathbb{N}$, what we are defining is well-defined and the mapping $k \mapsto [n+k, n]$ is thus independent of the natural number n. This is an isomorphism of rings since $[k+m, m] = [j+n, n]$ if and only if $k = j$. Likewise, $[k+n, n]+[j+m, m] = [k+j+m+n, m+n]$ and $[k+n, n]\cdot[j+m, m] = [(k+n)(j+m)+nm, (k+n)m+n(j+m)] = [jk+l, l]$ where $l = nj + km + 2nm$.

We define the zero of $\mathcal{Z}$ to be the element $[m, m]$, where $m \in \mathbb{N}$ is any fixed integer. Clearly $[j, k] + [m, m] = [j + k, k + m] = [j, k]$, $[j, k] \cdot [m, m] = [jm + km, jm + km] = [m, m]$. We define the negative of an element $[j, k]$ to be $[k, j]$ since $[j, k] + [k, j] = [j + k, k + j] = [m, m]$. From this we see that we can identify the collection of negatives of elements of $\mathcal{P}$ with the collection of elements of the form $[j, j + k]$, and such an element can be thought of as the negative of the natural number k. Through this, we identify the collection of integers, commonly denoted by $\mathbb{Z}$ with the integral domain $\mathcal{Z}$.

The Construction of the Rational Numbers from the Integers

In the future we will write a rational number in the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. We will define this collection from $\mathbb{Z} \times \mathbb{Z}$ by defining binary operations $+, \cdot$ from those on $\mathbb{Z}$.

Let

$$Q = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}.$$

We need to identify ordered pairs that would give the same value of $\frac{a}{b}$, where we think of (a, b) as $\frac{a}{b}$ or ab^{-1}. To do this, we define an equivalence relation $\sim$ on Q, which is given by

$$(a, b) \sim (c, d) \quad \iff ad = bc.$$

Clearly $(a, b) \sim (a, b)$ since $ab = ba$. Thus $\sim$ is reflexive.

$$(a, b) \sim (c, d) \iff ad = bc \iff da = cb \iff (c, d) \sim (a, b).$$

Thus we see $\sim$ is symmetric. Moreover, if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then we know $ad = cb$ and $cf = de$. Now,

$$ad = cb \Rightarrow adf = cbf$$

and

$$cf = de \Rightarrow cfb = deb.$$

Since $\mathbb{Z}$ is commutative we know

$$adf = deb,$$

or simply

$$(af)d = (be)d.$$

Since $d \neq 0$, by the cancellation property we may conclude that

$$af = be,$$

and thus

$$(a,b) \sim (e,f).$$

Hence we have shown $\sim$ is transitive.

We have established $\sim$ is an equivalence relation on Q, and hence we may view Q as a disjoint union of equivalence classes. Let $[a,b]$ denote the equivalence class of (a,b). Let $\mathcal{Q}$ be the set of equivalence classes. On this set we define $+, \cdot$ by:

$$[a,b] + [c,d] = [ad + bc, bd]$$

and

$$[a,b] \cdot [c,d] = [ac, bd].$$

To understand these definitions, simply remember the formulae:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

However, before we proceed any further, we must make certain these definitions of $+, \cdot$ on $\mathcal{Q}$ are well-defined. That is if $[a,b] = [r,s]$ and $[c,d] = [t,u]$, then

$$[ad + bc, bd] = [a,b] + [c,d]$$

$$= [r, s] + [t, u] = [ru + st, su]$$

and

$$[ac, bd] = [a, b] \cdot [c, d] = [r, s] \cdot [t, u] = [rt, su].$$

That is we actually need to show

$$[ad + bc, bd] = [ru + st, su]$$

and

$$[ac, bd] = [rt, su]$$

whenever $(a, b) \sim (r, s)$ and $(c, d) \sim (t, u)$.

Taking the condition $as = br$ and multiplying both sides by cu gives

$$ascu = brcu.$$

However, $cu = dt$ and thus

$$ascu = bdrt,$$

which implies

$$[ac, bd] = [rt, su].$$

Taking the condition $as = br$ and multiplying both sides by du gives

$$adsu = brdu.$$

Taking the condition $cu = dt$, and multiplying both sides by bs gives us

$$bcsu = dtbs.$$

Adding these together gives

$$(ad + bc)su = (ru + st)bd,$$

and thus

$$[ad + bc, bd] = [ru + st, su].$$

Thus we see that the binary operations $+, \cdot$ on $\mathcal{Q}$ are well-defined. It is left as an exercise to check that addition and multiplication are associative and commutative, and the distributive property holds.

In $\mathcal{Q}$ the element $[0,1]$ is the zero element, and the element $[1,1]$ is the unity. The negative of $[a,b]$ is $[-a,b]$ and the multiplicative inverse of a non-zero element $[a,b]$ is $[b,a]$. Thus we see that $\mathcal{Q}$ is a field. Moreover, we can identify $\mathbb{Z}$ with the subset $\{[a,1] : a \in \mathbb{Z}\}$ and it is clear that we may view $\mathcal{Q}$ as an extension of $\mathbb{Z}$, since

$$[a,1] + [b,1] = [a+b,1]$$

and

$$[a,1] \cdot [b,1] = [ab,1].$$

Moreover, any non-zero $a \in \mathbb{Z}$ has $[a,1]$ as an invertible element in $\mathcal{Q}$.

This field $(\mathcal{Q}, +, \cdot)$ is what we usually called the field of rational numbers $(\mathbb{Q}, +, \cdot)$. We instead use familiar the notation $\frac{a}{b}$ for $[a,b]$.

Order on the Set of Rational Numbers

We may define an ordering on the set of integers, which extends to the set of rational numbers as follows: The **positive** integers are the natural numbers, and the **negative** integers are the integers whose additive inverses are positive. The intersection between these two collections is empty. We say that an integer is **non-negative** if it is positive or zero, and **non-positive** if it is negative or zero.

Note that from its definition, and the property that $-n = (-1) \cdot n$ holds for any $n \in \mathbb{Z}$, as well as $(-1)^2 = 1$, we get that the set of positive integers are closed under $\cdot$, a positive times a negative is negative, and a negative times a negative is a positive.

We extend the notions of positivity and negativity to set of the rational numbers. We say that a rational number $\frac{m}{n}$ is positive if m, n are either both positive or both negative. A rational number $\frac{m}{n}$ is negative if one of $\{m, n\}$ is positive, and the other is negative. From these definitions, the properties of the set of positive rational numbers and negative rational numbers satisfy: the positive rational numbers are closed under $\cdot$, a positive times a negative is negative, and a negative times a negative is a positive.