Disproof of Bell's Theorem

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Illuminating the Illusion of Entanglement

Second Edition

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Illuminating the Illusion of Entanglement

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To

Johannes Kepler,

who dared to consider ellipse.

I cannot seriously believe in [quantum mechanics] because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance.

Albert Einstein

Contents

Preface to Second Edition

In this edition I have included two new chapters, Chapters 9 and 10, bringing the total number of chapters to 12. Both chapters are primarily my replies to specific criticisms, but they also bring fresh and deeper insights into the main objective of the book—which is to understand the origins of quantum correlations. In this regard the appendix after Chapter 9 is of special interest, because it includes a discussion of two event-by-event simulations of my 3-sphere model for the EPR correlation, confirming it beyond any shadow of doubt. I am indebted to Chantal Roth and Michel Fodje for the simulations.

Unfortunately there are some further results in the subject that I have not been able to include in this edition. Some of these results can be found on my blog at http://libertesphilosophica.info/blog/, which is dedicated to discussions about the contents of this book.

The work on the two new chapters included in this edition was funded by a grant from the Foundational Questions Institute (FQXi) Fund, a donor advised fund of the Silicon Valley Community Foundation on the basis of proposal FQXi-MGA-1215 to the Foundational Questions Institute. It was carried out while I was an affiliate of Theiss Research, USA, and of Wolfson College and the Department of Materials of the University of Oxford, UK. I am grateful to Martin Castell for his continued hospitality in the Department of Materials during the course of this work, and to Fred Diether and Tom Ray for their moral support in the last few years of my work on this subject.

A special thanks goes to my friend Lucien Hardy, who has spent considerable amount of his valuable time trying to understand the ideas discussed in this book. Finally, I thank the readers of the first edition of the book for their comments and words of encouragement.

Joy J. Christian
Oxford, England
December, 2013

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Over the years I have benefited from discussions and debates about Bell's theorem with many friends and colleagues. My debt to them has been duly acknowledged individually at the end of each paper collected here. There are some names, however, that are either missing from these acknowledgements or deserve special mention. Among these, I am particularly grateful to Daniel ben-Avraham, David Coutts, Azhar Iqbal, Daniel Rohrlich, Michael Seevinck, Jeff Uhlmann, and Gregor Weihs for discussions and criticisms.

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Finally I wish to thank Jeff Young for his infinite patience with my slow pace, Shereen Siddiqui for the beautiful cover design of this book, and Christie Mayer for her masterful editorial assistance.

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x x x

In a good mystery story the most obvious clues often lead to the wrong suspects.

Einstein and Infeld

x x x

Chapter **1**

On the Local-Realistic Origins of Quantum Correlations

Abstract: It is well known that quantum correlations are not only more disciplined (and hence stronger) compared to classical correlations, but they are more disciplined in a mathematically very precise sense. This raises an important physical question: What is responsible for making quantum correlations so much more disciplined? Here we explain the observed discipline of quantum correlations by identifying the symmetries of our physical space with those of a parallelized 7-sphere. We substantiate this identification by proving that any quantum correlation can be understood as a classical, local-realistic correlation among a set of points of a parallelized 7-sphere.

1.1 Introduction

In 1927, when quantum theory was still in its infancy and John Bell was yet to be born, Albert Einstein—one of the founding fathers of the theory—was attending the now famous 5^{th} Solvay Conference in Brussels. He was profoundly disturbed by what the new theory had to say about the nature of physical reality. Among other things, his concerns stemmed from a deep appreciation of unity in nature. Beyond the cliché of "God does not play dice", he had recognized that quantum theory entailed a fundamental schism in nature. The aphorism “God does not split reality” perhaps better captures the true essence of his concerns [1]. What he sought was a unified picture of nature, devoid of any subjective boundary between the classical and the quantum. What he suspected was a deeper layer of reality, beyond the polarized picture offered by quantum theory.

By 1935—when John Bell was seven—these worries of Einstein had matured into a powerful logical argument against the new theory. Published in collaboration with Boris Podolsky and Nathan Rosen, this argument proves, *once and for all*, that quantum theory provides at best an incomplete description of the physical reality [2]. Since the argument itself is logically impeccable, this conclusion is beyond dispute. Any argument, however, can only be as good as its premises, and that—as is well known—is where Bell entered the game in 1964 [3]. He attempted to show that not all of the premises of EPR are mutually compatible. Ironically, however, it is the argument of Bell that turns out to contain a faulty assumption, not that of EPR. What is more, this assumption appears in the very first equation of Bell's famous paper [3], and yet it had escaped notice until recently.

In a series of papers, written between 2007 and 2013, I tried to bring out Bell's error and constructed explicit counterexamples, not only to his original theorem, but also to several of its variants. This book is a collection of these papers, each of which can be read more or less independently, but their contents are interconnected, and reveal different aspects of the fundamental flaw in Bell's argument. The collection as a whole, however, is better viewed as addressing a very important physical question. Regardless of the validity of his theorem, what Bell discovered in 1964 is physically quite significant. He discovered that quantum correlations are far more disciplined than any classically possible correlation. What is more, quantum correlations are not only more disciplined, but are more disciplined in a mathematically very precise sense. This tells us something much more profound about the structure of the world we live in. And, at the same time, it raises a very important physical question:

What is it that makes quantum correlations more disciplined than classical correlations?

My goal in this book is to answer this question in mathematically and physically as precise a sense as possible. To this end, let me begin with an extended summary of my argument against Bell's theorem.

As noted above, the story began with Einstein, Podolsky, and

Rosen [2]. The logic of their argument can be summarized as follows:

$$
\begin{aligned}
&\textbf{(1)}\ \text{QM} \Longrightarrow \text{Perfect Correlations}\\
+\ &\textbf{(2)}\ \text{Adherence to Local Causality}\\
+\ &\textbf{(3)}\ \text{Criterion of Objective Reality}\\
+\ &\textbf{(4)}\ \text{Notion of a Complete Theory}\\
\Longrightarrow\ &\textbf{(5)}\ \text{QM is an Incomplete Theory.}
\end{aligned}
$$

Given their premises, the conclusion of EPR follows impeccably. Among their premises (which are hardly unreasonable), the one that concerns us the most is their criterion of completeness:

every element of the physical reality must have a counterpart in the physical theory.

Bell attempted to prove that no theory satisfying this criterion can be locally causal. To this end, he took a complete theory to mean any theory whose predictions are dictated by functions of the form

$$\mathscr{A}(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^0 \equiv \{-1,\,+1\}, \tag{1.1}$$

where $\mathrm{I\!R}^3$ is the space of 3-vectors, Λ is a space of "complete" states, and $S^0 \equiv \{-1,\,+1\}$ is a unit 0-sphere. He then claimed (correctly, as it turns out) that no pair of functions of this form can reproduce the correlation for the singlet state predicted by quantum mechanics[1]:

$$\langle\, \mathscr{A}(\mathbf{a},\,\lambda)\, \mathscr{B}(\mathbf{b},\,\lambda)\,\rangle \,\neq\, -\,\mathbf{a}\cdot\mathbf{b}\,. \tag{1.2}$$

At first sight, this appears to be a straightforward mathematical contradiction undermining the force of the EPR argument [3]. And for this reason functions of this form are routinely assumed in the Bell literature to provide complete specifications of the elements of physical reality, or complete accounting of all possible measurement results. As we shall see however, Bell's prescription is not only false, it is breathtakingly naïve and unphysical. It stems from an incorrect underpinning of both the EPR argument and the actual topological configurations involved in the relevant experiments [4]. In truth, for

[1] This is hardly surprising. After all, the product moment correlation coefficient employed by Bell in his paper, by definition, is a measure of *linear* relationship between bivariate variables. Thus Bell implicitly assumed a linear relationship between $\mathscr{A}$ and $\mathscr{B}$ to prove that the relationship between them must be linear!

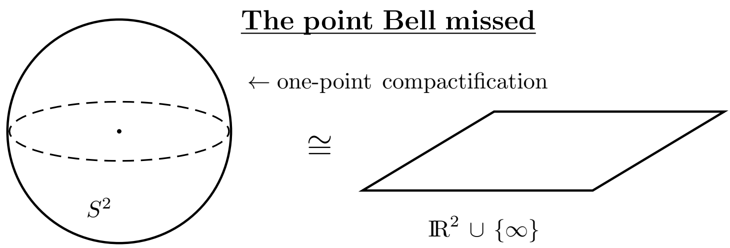

Figure 1.1: Although lines and planes contain the same number of points, it is impossible to put the points of a line or a plane in a one-to-one correspondence with all of the points of a 2-sphere.

any two-level system the EPR criterion of completeness demands that the correct functions must necessarily be of the form

$$\pm 1 \;=\; \mathscr{A}(\mathbf{n},\, \lambda) : \mathbb{R}^3 \times \Lambda \longrightarrow S^3 \hookrightarrow \mathbb{R}^4, \tag{1.3}$$

with the *simply-connected* codomain S^3 of $\mathscr{A}(\mathbf{n},\, \lambda)$ replacing the *totally disconnected* codomain S^0 assumed by Bell. It is important to note here that this correction does not affect the actual measurement results. For a specific vector $\mathbf{n}$ and an initial state λ we still have

$$\mathscr{A}(\mathbf{n},\, \lambda) \;=\; +1 \;\text{ or }\; -1 \tag{1.4}$$

as demanded by Bell, but now the topology of the codomain of the function $\mathscr{A}(\mathbf{n},\, \lambda)$ has changed from a 0-sphere to a 3-sphere, with the latter embedded in $\mathbb{R}^4$ in such a manner that the above constraint is satisfied. On the other hand, as is evident from Fig. 1.1 (and will be further clarified in the following pages), without this topological correction it is impossible to provide a complete account of all possible measurement results. Thus the selection of the codomain $S^3 \hookrightarrow \mathbb{R}^4$ in equation (1.3) is not a matter of choice but necessity. What is responsible for the EPR correlations is *the topology of the set of all possible measurement results* [4]. And for a two-level system this set happens to be an equatorial 2-sphere within a parallelized 3-sphere. But once the codomain of the functions $\mathscr{A}(\mathbf{n},\, \lambda)$ is so corrected, the proof of Bell's theorem (as given in Ref. [3]) simply falls apart. In fact, as we shall repeatedly see in the following pages, the strength of the correlation for *any* physical system is entirely determined by the topology of the codomain of the local functions $\mathscr{A}(\mathbf{n},\, \lambda)$. It has nothing whatsoever to do with entanglement or nonlocality.

Put differently, once the measurement results are represented by functions of the form (1.3), it is quite easy to reproduce the quantum correlations purely classically, in a manifestly local-realistic manner. For example, suppose Alice and Bob are equipped with the variables

$$\mathscr{A}(\mathbf{a},\,\lambda)\;=\;\{-\,a_j\;\boldsymbol{\beta}_j\,\}\,\{\,a_k\;\boldsymbol{\beta}_k(\lambda)\,\}\;=\;\begin{cases}+\,1 & \text{if}\quad \lambda\,=\,+\,1\\ -\,1 & \text{if}\quad \lambda\,=\,-\,1\end{cases} \tag{1.5}$$

and

$$\mathscr{B}(\mathbf{b},\,\lambda)\;=\;\{+\,b_k\;\boldsymbol{\beta}_k\,\}\,\{\,b_j\;\boldsymbol{\beta}_j(\lambda)\,\}\;=\;\begin{cases}-\,1 & \text{if}\quad \lambda\,=\,+\,1\\ +\,1 & \text{if}\quad \lambda\,=\,-\,1\,,\end{cases} \tag{1.6}$$

where the repeated indices are summed over x, y, and z; the fixed bivector basis $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$ is defined by the algebra

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\;=\;-\,\delta_{jk}\;-\;\epsilon_{jkl}\,\boldsymbol{\beta}_l\,; \tag{1.7}$$

and—together with $\boldsymbol{\beta}_j(\lambda)=\lambda\,\boldsymbol{\beta}_j$—the λ-dependent bivector basis $\{\boldsymbol{\beta}_x(\lambda),\,\boldsymbol{\beta}_y(\lambda),\,\boldsymbol{\beta}_z(\lambda)\}$ is defined by the algebra

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\;=\;-\,\delta_{jk}\;-\;\lambda\;\epsilon_{jkl}\,\boldsymbol{\beta}_l\,, \tag{1.8}$$

where $\lambda\;=\;\pm\,1$ is a fair coin representing two alternative orientations of the 3-sphere[2], δ_{jk} is the Kronecker delta, ϵ_{jkl} is the Levi-Civita symbol, and $\mathbf{a}=a_j\,\mathbf{e}_j$ and $\mathbf{b}=b_j\,\mathbf{e}_j$ are unit vectors [5]. Evidently, the variables $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ belonging to S^3—in addition to being manifestly realistic—are strictly *local* variables. In fact, they are not even contextual [6]. Alice's measurement result—although it refers to a freely chosen direction **a**—depends only on the initial state λ; and likewise, Bob's measurement result—although it refers to a freely chosen direction **b**—depends only on the initial state λ.

In the subsequent chapters we shall mainly use the standard notations of Clifford algebra $Cl_{3,0}$. The bivector algebras (1.7) and (1.8) will then be seen as even subalgebras of $Cl_{3,0}$. The latter is a linear vector space, $\mathrm{I\!R}^8$, spanned by the graded orthonormal basis

$$\{1,\;\; \mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z,\;\; \mathbf{e}_x\wedge\mathbf{e}_y,\,\mathbf{e}_y\wedge\mathbf{e}_z,\,\mathbf{e}_z\wedge\mathbf{e}_x,\;\; \mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\}, \tag{1.9}$$

[2] Needless to say, $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ are two *different* functions of the random variable λ. Moreover, they are *statistically independent events* occurring within a 3-sphere, with factorized joint probability $P(\mathscr{A}\text{ and }\mathscr{B})=P(\mathscr{A})\times P(\mathscr{B})\leq\frac{1}{2}$. Therefore their product $\mathscr{A}\mathscr{B}$ is guaranteed to be equal to -1 only for the case $\mathbf{a}=\mathbf{b}$. For all other **a** and **b**, $\mathscr{A}\mathscr{B}$ will alternate between the values -1 and $+1$.

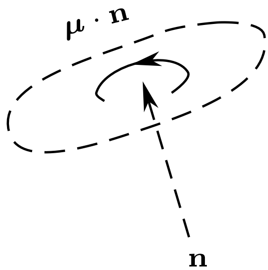

Figure 1.2: A unit bivector represents an equatorial point of a unit, parallelized 3-sphere. It is an abstraction of a directed plane segment, with only a magnitude and a sense of rotation—*i.e.*, clockwise (−) or counterclockwise (+). Neither the depicted oval shape of its plane, nor its axis of rotation $\mathbf{n}$, is an intrinsic part of the bivector $\boldsymbol{\mu}\cdot\mathbf{n}$.

where "∧" is the outer product, and the trivector $I \equiv \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z$ defines the fundamental volume form of the physical space. In terms of these notations we can rewrite the bivector $\{\, a_j\, \boldsymbol{\beta}_j(\lambda)\,\}$ as

$$\boldsymbol{\mu}\cdot\mathbf{a} \,\equiv\, \{\, a_j\; \boldsymbol{\beta}_j(\lambda)\,\} \,\equiv\, \lambda\,\{\, a_x\; \mathbf{e}_y \,\wedge\, \mathbf{e}_z \,+\, a_y\; \mathbf{e}_z \,\wedge\, \mathbf{e}_x \,+\, a_z\; \mathbf{e}_x \,\wedge\, \mathbf{e}_y\}, \tag{1.10}$$

with $\boldsymbol{\mu} = \lambda\, I$ now representing the hidden variable of the model. The variables $\mathscr{A}(\mathbf{a},\, \lambda)$ and $\mathscr{B}(\mathbf{b},\, \lambda)$ defined above then take the form

$$S^3 \ni \mathscr{A}(\mathbf{a},\, \boldsymbol{\mu}) \,=\, (-\, I\cdot\mathbf{a}\,)\,(\,+\,\boldsymbol{\mu}\cdot\mathbf{a}\,) \,=\, \begin{cases} +\,1 & \text{if} \quad \boldsymbol{\mu} \,=\, +\,I \\ -\,1 & \text{if} \quad \boldsymbol{\mu} \,=\, -\,I \end{cases} \tag{1.11}$$

and

$$S^3 \ni \mathscr{B}(\mathbf{b},\, \boldsymbol{\mu}) \,=\, (+\, I\cdot\mathbf{b}\,)\,(\,+\,\boldsymbol{\mu}\cdot\mathbf{b}\,) \,=\, \begin{cases} -\,1 & \text{if} \quad \boldsymbol{\mu} \,=\, +\,I \\ +\,1 & \text{if} \quad \boldsymbol{\mu} \,=\, -\,I, \end{cases} \tag{1.12}$$

with the trivector $\boldsymbol{\mu}$ being either $+\,I$ or $-\,I$ with equal probability. In what follows we shall view the fixed bivectors $(-\,I\cdot\mathbf{a}\,)$ and $(+\,I\cdot\mathbf{b}\,)$ as representing the measuring instruments for detecting the random bivectors $(\,+\,\boldsymbol{\mu}\cdot\mathbf{a}\,)$ and $(\,+\,\boldsymbol{\mu}\cdot\mathbf{b}\,)$, which represent the spins.

It is crucial to note that the variables $\mathscr{A}(\mathbf{a},\, \lambda)$ and $\mathscr{B}(\mathbf{b},\, \lambda)$ are generated with *different* bivectorial scales of dispersion (or different standard deviations) for each direction $\mathbf{a}$ and $\mathbf{b}$. Consequently, in statistical terms these variables are raw scores, as opposed to standard scores [7]. Recall that a standard score indicates how many standard deviations an observation or datum is above or below the mean. If x is a raw (or unnormalized) score and $\overline{\mathrm{x}}$ is its mean value, then the standard (or normalized) score, z(x), is defined by

$$\mathrm{z}(\mathrm{x}) = \frac{\mathrm{x} - \overline{\mathrm{x}}}{\sigma(\mathrm{x})}\,, \tag{1.13}$$

where $\sigma(\mathrm{x})$ is the standard deviation of x. A standard score thus represents the distance between a raw score and the population mean in the units of standard deviation, and allows us to make comparisons of raw scores that may have come from very different sources. In other words, the mean value of the standard score itself is always zero, with standard deviation unity. In terms of these concepts the bivariate correlation between raw scores x and y is defined as

$$\mathcal{E}(\mathrm{x},\,\mathrm{y}) = \frac{\lim\limits_{n\,\gg\,1}\left[\frac{1}{n}\sum\limits_{i\,=\,1}^{n}(\mathrm{x}^i - \overline{\mathrm{x}})\,(\mathrm{y}^i - \overline{\mathrm{y}})\right]}{\sigma(\mathrm{x})\,\sigma(\mathrm{y})} \tag{1.14}$$

$$= \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\mathrm{z}(\mathrm{x}^i)\,\mathrm{z}(\mathrm{y}^i)\right]. \tag{1.15}$$

It is vital to appreciate that covariance by itself—*i.e.*, the numerator of equation (1.14) by itself—does not provide the correct measure of association between the raw scores, not the least because it depends on different units and scales (or different scales of dispersion) that may have been used (advertently or inadvertently) in the measurements of such scores [7]. Therefore, to arrive at the correct measure of association between the raw scores one must either use equation (1.14), with the product of standard deviations in the denominator, or use covariance of the standardized variables, as in Eq. (1.15).

These basic statistical concepts are crucial for understanding the EPR correlations. As defined above, the random variables $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ are products of two factors—one random and another non-random. Within $\mathscr{A}(\mathbf{a},\,\lambda)$ the factor $\{\,a_k\,\boldsymbol{\beta}_k(\lambda)\,\}$ is a random factor—a function of the hidden variable λ, whereas $\{-\,a_j\,\boldsymbol{\beta}_j\,\}$ is a non-random factor, independent of the hidden variable λ. Thus, as a random variable each number $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ is generated with a *different* standard deviation—*i.e.*, a *different* size of typical error. More specifically, $\mathscr{A}(\mathbf{a},\,\lambda)$ is generated with the standard deviation $\{-\, a_j\, \boldsymbol{\beta}_j\,\}$, whereas $\mathscr{B}(\mathbf{b},\,\lambda)$ is generated with a different standard deviation, namely $\{+\, b_k\, \boldsymbol{\beta}_k\,\}$. These two deviations can be calculated easily. Since errors in linear relations propagate linearly, the standard deviation of $\mathscr{A}(\mathbf{a},\,\lambda)$ is equal to $\{-\, a_j\, \boldsymbol{\beta}_j\,\}$ times the standard deviation of $\{\, a_k\, \boldsymbol{\beta}_k(\lambda)\,\}$ (which we write as $\sigma(A)$), whereas the standard deviation of $\mathscr{B}(\mathbf{b},\,\lambda)$ is equal to $\{+\, b_k\, \boldsymbol{\beta}_k\,\}$ times the standard deviation of $\{\, b_j\, \boldsymbol{\beta}_j(\lambda)\,\}$ (which we write as $\sigma(B)$):

$$\begin{aligned} \sigma(\mathscr{A}\,) &= \{-\, a_j\, \boldsymbol{\beta}_j\,\}\,\sigma(A) \\ \text{and}\;\; \sigma(\mathscr{B}\,) &= \{+\, b_k\, \boldsymbol{\beta}_k\,\}\,\sigma(B). \end{aligned} \tag{1.16}$$

But since all the bivectors we have been considering are normalized to unity, and since the mean of $\{\, a_k\, \boldsymbol{\beta}_k(\lambda)\,\}$ vanishes on the account of λ being a fair coin, its standard deviation is easy to calculate, and it turns out to be equal to unity:

$$\begin{aligned} \sigma(A) &= \sqrt{\frac{1}{n}\sum_{i\,=\,1}^{n} \Big|\Big|\, A(\mathbf{a},\,\lambda^i) \,-\, \overline{A(\mathbf{a},\,\lambda^i)}\,\Big|\Big|^2} \\ &= \sqrt{\frac{1}{n}\sum_{i\,=\,1}^{n} ||\,\{\, a_k\, \boldsymbol{\beta}_k(\lambda^i)\,\} \,-\, 0\,||^2} \;=\; 1, \end{aligned} \tag{1.17}$$

where the last equality follows from the fact that $\{\, a_k\, \boldsymbol{\beta}_k(\lambda^i)\,\}$ are normalized to unity. Similarly, we find that $\sigma(B)$ is also equal to 1. As a result, the standard deviation of $\mathscr{A}(\mathbf{a},\,\lambda)$ works out to be equal to $\{-\, a_j\, \boldsymbol{\beta}_j\,\}$, and the standard deviation of $\mathscr{B}(\mathbf{b},\,\lambda)$ works out to be equal to $\{+\, b_k\, \boldsymbol{\beta}_k\,\}$. Putting these two results together, we arrive at the following standardized scores corresponding to the raw scores:

$$\begin{aligned} A(\mathbf{a},\,\lambda) &= \frac{\mathscr{A}(\mathbf{a},\,\lambda) \,-\, \overline{\mathscr{A}(\mathbf{a},\,\lambda)}}{\sigma(\mathscr{A})} \\ &= \frac{\mathscr{A}(\mathbf{a},\,\lambda) \,-\, 0}{\{-\, a_j\, \boldsymbol{\beta}_j\,\}} \;=\; \{\, a_k\, \boldsymbol{\beta}_k(\lambda)\,\} \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} B(\mathbf{b},\,\lambda) &= \frac{\mathscr{B}(\mathbf{b},\,\lambda) \,-\, \overline{\mathscr{B}(\mathbf{b},\,\lambda)}}{\sigma(\mathscr{B})} \\ &= \frac{\mathscr{B}(\mathbf{b},\,\lambda) \,-\, 0}{\{+\, b_k\, \boldsymbol{\beta}_k\,\}} \;=\; \{\, b_j\, \boldsymbol{\beta}_j(\lambda)\,\}, \end{aligned} \tag{1.19}$$

where we have used the identities such as $\{+\, a_k\, \boldsymbol{\beta}_k\,\}\{-\, a_k\, \boldsymbol{\beta}_k\,\} = +1$. Not surprisingly, just like the raw scores $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$, these

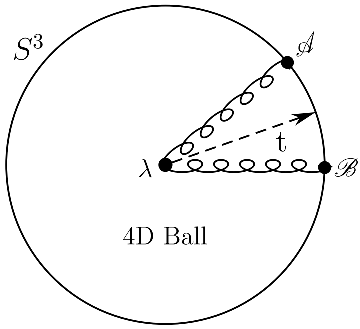

Figure 1.3: An initial EPR state λ originated at time $\mathrm{t} = 0$ evolves into measurement results $\mathscr{A}$ and $\mathscr{B}$ at a later time, occurring at two spacelike separated locations on a parallelized 3-sphere, S^3, which can be thought of as a boundary of a 4-dimensional ball of radius t. In what follows we shall assume that t has been normalized to unity.

standard scores are also strictly *local* variables: $A(\mathbf{a},\,\lambda)$ depends only on the freely chosen local direction $\mathbf{a}$ and the common cause λ, and likewise $B(\mathbf{b},\,\lambda)$ depends only on the freely chosen local direction $\mathbf{b}$ and the common cause λ. Moreover, despite appearances, $A(\mathbf{a},\,\lambda)$ and $B(\mathbf{b},\,\lambda)$ are simply binary numbers, $\pm\,1$, albeit occurring within the compact topology of the 3-sphere rather than the real line:

$$S^3 \supset S^2 \ni A(\mathbf{a},\,\lambda) \,=\, \{\, a_k\; \boldsymbol{\beta}_k(\lambda)\,\} \,=\, \pm\,1 \;\text{ about } \mathbf{a} \in \mathrm{I\!R}^3, \tag{1.20}$$

$$S^3 \supset S^2 \ni B(\mathbf{b},\,\lambda) \,=\, \{\, b_j\; \boldsymbol{\beta}_j(\lambda)\,\} \,=\, \pm\,1 \;\text{ about } \mathbf{b} \in \mathrm{I\!R}^3. \tag{1.21}$$

In fact, since the space of all bivectors $\{\, a_k\; \boldsymbol{\beta}_k(\lambda)\,\}$ is isomorphic to the equatorial 2-sphere contained within the 3-sphere [4], each standard score $A(\mathbf{a},\,\lambda)$ of Alice is uniquely identified with a definite point of this 2-sphere, and likewise for the standard scores of Bob.

> **In the following chapters we shall tacitly assume that this procedure of standardizing from the raw scores to standard scores has been performed for all measurement results, taken either as points of a 3-sphere, or more generally as points of a 7-sphere.**

Now, since we have assumed that initially there was 50/50 chance between the right-handed and left-handed orientations of the 3-sphere